Macroeconomic Analysis

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Chapter 1

Microeconomic Foundations

Modern macroeconomic models are micro founded and dynamic. They are built on assumptions about microeconomic primitives, specifically preferences and technology; explicitly model intertemporal links and expectation formation; and describe economic outcomes as the result of optimizing choices by households and firms, over time, subject to affordability and feasibility constraints. This contrasts with frameworks such as the IS-LM model which posit macroeconomic relationships, for example between consumption and income, and often are static. Modern macroeconomics uses the concepts and tools of *general equilibrium* theory; unlike *partial equilibrium* analysis of a single market, general equilibrium analysis accounts for feedback effects across markets.

To prepare for the subsequent micro founded macroeconomic analysis, we review key microeconomic concepts from general equilibrium theory and introduce assumptions about primitives.

1.1 Microeconomics

1.1.1 Allocation, Feasibility, Optimality

An *allocation* consists of a *consumption* vector for each household and a net *production* vector for each firm. For example, an allocation in an economy with one household, one firm and three goods could be $\{(1,2,1),(-1,1,-1)\}$: The household consumes one unit each of the first and third good and two units of the second good, while the firm uses one unit each of the first and third good as inputs and supplies one unit of the second good. In a model with government the allocation also includes a consumption and production vector for the government. In an open economy model the allocation also includes a consumption and production vector for the rest of the world.

An allocation is *feasible* if for each good, total consumption does not exceed the endowment plus net production. For example, the allocation given above is feasible if the endowment vector equals $(2,1,2)$ but it is not feasible if the endowment vector equals $(2,2,1)$.

A feasible allocation Pareto dominates another feasible allocation if at least one household strictly prefers the former and no household strictly prefers the latter. A feasible allocation is *Pareto optimal* or Pareto efficient if it is not Pareto dominated by any other feasible allocation. The set of Pareto optimal allocations traces the Pareto frontier.

1.1.2 Competitive Equilibrium

The consumption set of a household contains all consumption vectors that the household conceivably could consume in the absence of budgetary restrictions. For example, the consumption set might exclude negative quantities of apples.

The budget set of a household contains all consumption vectors in the household's consumption set that the household can afford to consume. The budget set is determined by household endowments, the prices of all goods, and firm profits which are distributed to households according to prespecified ownership rights.

The production set of a firm contains all production vectors that are feasible given the firm's technology.

A *competitive equilibrium* or *Walrasian equilibrium* is an allocation and a set of prices satisfying three conditions, conditional on endowments, firm production sets, and household preferences:

i. The allocation is feasible.

ii. Taking prices as given, each firm's production choice is profit maximizing.

iii. Taking prices and firm profits as given, each household's consumption choice is utility maximizing in the household's budget set.

The equilibrium is competitive because firms and households take prices as given. Alternative, non-competitive equilibria might exist as well where agents perceive their choices to affect prices or firm profits, and they exploit this feature. For example, a firm might want to reduce output in order to raise the equilibrium price of its product. We mostly abstract from non-competitive behavior and focus on competitive equilibria.

A competitive equilibrium with lump-sum transfers between households that sum to zero is referred to as price equilibrium with transfers.

1.1.3 Walras' Law

Let p denote the vector of prices across goods. Let $z_g^h(p)$ denote household h's *net demand function* for good g—the household's desired consumption net of its endowment and share of firm profits—as a function of p. Let $z^h(p)$ denote the vector of household h's net demand functions across goods. Let $z_g(p) \equiv \sum_h z_g^h(p)$ denote the aggregate excess demand function for good g. Finally, let $z(p) \equiv \sum_h z^h(p)$ denote the vector of excess demand functions across goods. If all households satisfy their budget constraints, then $p \cdot z^h(p) = 0$ for all h. By implication, *Walras' Law* holds: The values of excess demands sum to zero, $p \cdot z(p) = 0$.

Walras' Law has two important consequences. First, in an equilibrium with strictly positive prices all markets clear. To see this, note that the equilibrium requirements optimization and feasibility (subject to free disposal) imply $z(p) \leq 0$. If a good has strictly positive price, excess demand for that good therefore must be zero (otherwise, $p \cdot z(p) \neq 0$). Second, with strictly positive prices, market clearing in all markets but one implies market clearing in the remaining market. To see this, suppose all markets except market j clear, $z_g(p) = 0$ for all $g \neq j$, such that $\sum_{g \neq j} p_g z_g(p) = 0$. Since $p_j > 0$, $z_j(p)$ also must equal zero (otherwise, $p \cdot z(p) \neq 0$).

1.1.4 Fundamental Theorems of Welfare Economics

The fundamental theorems of welfare economics relate equilibrium allocations and Pareto optimal allocations.

The *first fundamental theorem of welfare economics* formalizes the notion of an *invisible hand*. It states that, if an allocation and price system constitute a price equilibrium with transfers (in particular, a competitive equilibrium) and certain conditions are satisfied, then the allocation is Pareto optimal. Decentralized choices by price taking individuals thus are consistent with Pareto optimality, and this holds true even if lump-sum transfers occur before market transactions take place.

Let x^h, e^h, and y^f denote household h's consumption and endowment vectors as well as firm f's net production vector, respectively. An allocation $\{\{x^h\}_h, \{y^f\}_f\}$ is feasible if $\sum_h (x^h - e^h) \leq \sum_f y^f$. Let $\{\{x^{h\star}\}_h, \{y^{f\star}\}_f, p^\star\}$ be a competitive equilibrium with equilibrium price vector $p^\star$. The theorem claims that no feasible allocation Pareto dominates $\{\{x^{h\star}\}_h, \{y^{f\star}\}_f\}$. The proof by contradiction proceeds in three steps:

i. Suppose that a feasible Pareto dominating allocation, $\{\{x^{h\bullet}\}_h, \{y^{f\bullet}\}_f\}$, exists such that some household strictly prefers the consumption vector in the $\bullet$ allocation over the vector in the $\star$ allocation. Impose the condition that preferences are locally non-satiated; households and firms are competitive; and markets are complete (all goods are traded). Since the household chose optimally, the consumption vector in the $\bullet$ allocation then must be unaffordable under $p^\star$.

ii. Impose the condition that the market value of endowments be finite, for example because the number of households is finite. Summing over all households then implies
$$\sum_h p^\star \cdot (x^{h\bullet} - e^h) > \sum_f p^\star \cdot y^{f\star}.$$

iii. Since firms maximize profits, $\sum_f p^\star \cdot y^{f\star} \geq \sum_f p^\star \cdot y^{f\bullet}$. Combining the inequalities yields
$$\sum_h p^\star \cdot (x^{h\bullet} - e^h) > \sum_f p^\star \cdot y^{f\bullet}.$$

   But feasibility of $\{\{x^{h\bullet}\}_h, \{y^{f\bullet}\}_f\}$ implies the reverse inequality. We have therefore arrived at a contradiction.

The *second fundamental theorem of welfare economics* formalizes the notion that Pareto optimal allocations can be decentralized through markets and lump-sum transfers. It states that for every Pareto optimal allocation, there exists a set of prices such that the allocation and the prices constitute a price equilibrium with transfers. Necessary assumptions of the theorem include convex production sets as well as convex and locally non-satiated preferences.

1.2 Primitives

The *primitives* of a modern macroeconomic model—the objects we take as given—include those of a microeconomic model, most importantly households with preferences and endowments, and firms with technologies. An event tree serves to describe the dynamic and stochastic formation of the economy.

1.2.1 Event Tree

Time is denoted by t. It runs from zero, the initial date, to some final date T or to infinity. To represent exogenous risk we let ϵ^t denote the *history* of realizations of the *state of nature* up to and including date t. From the perspective of date $t = 0$, history ϵ^0 is known but history $\epsilon^t, t \geq 1$, is a random variable unless there is no risk.

The *event tree* in figure 1.1 illustrates a three-period example. At date $t = 1$, one of two possible realizations occurs, "up" (for example promotion) or "down" (demotion). The same happens at date $t = 2$. History ϵ^1 thus takes two values, (up) or (down), and history ϵ^2 four, (up, up), (up, down), (down, up), or $(\text{down}, \text{down})$.

Except for deterministic settings, variables need to be indexed by history to avoid ambiguity. Consider for instance a variable c, consumption of fruit say. Indexing c by ϵ^t accounts for the fact that fruit in different histories at the same date constitute different commodities, and that the quantities of fruit consumed in different histories may differ. In the environment of figure 1.1, fruit consumption at date $t = 2$ can take four values, $c_2(\text{up}, \text{up})$, $c_2(\text{up}, \text{down})$, $c_2(\text{down}, \text{up})$, or $c_2(\text{down}, \text{down})$.

A variable may be indexed by date t and history $\epsilon^s, s < t$. This indicates that the variable takes the same value across all histories at date t that are continuation histories of history ϵ^s. For example, in the environment of figure 1.1, $c_2(\text{up})$ would indicate that fruit consumption at date $t = 2$ in history (up, up) and in history (up, down) necessarily is the same.

1.2.2 Preferences

In the simplest dynamic model, *households* consume a single good (which might represent a composite), c, in each history. They trade off consumption across time and histories, in parallel to the static trade-off between apples and oranges, say, in a simple microeconomic model. As we will see in chapters 2 and 4, the optimality conditions

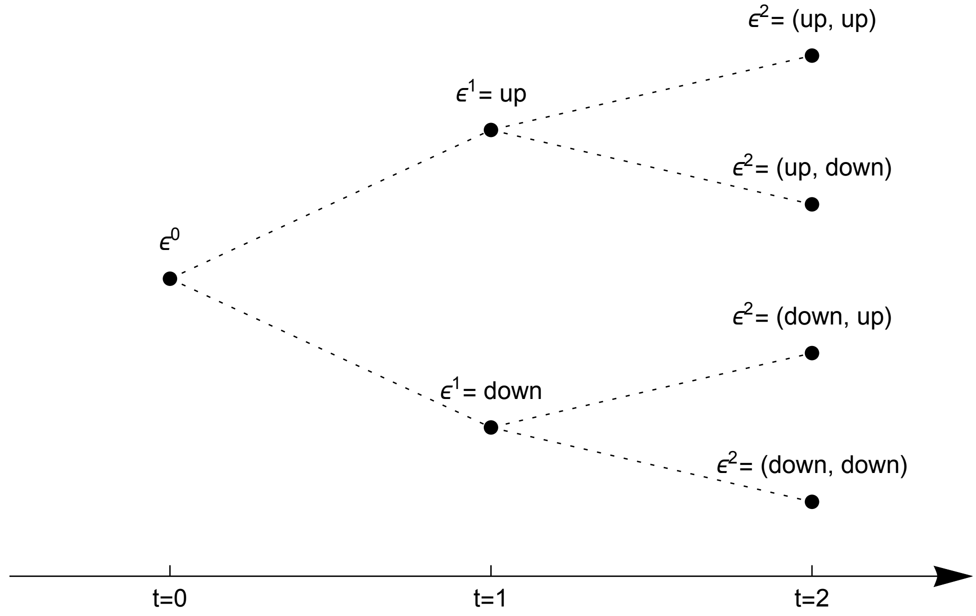

Figure 1.1: Event tree: An economy with three periods and two states of nature each at dates $t = 1, 2$.

in dynamic and static settings are isomorphic under specific assumptions about the market structure.

Preferences map a sequence of history-contingent consumption into utility,

$$U\left(c_0, \{c_1(\epsilon^1)\}_{\epsilon^1}, \ldots, \{c_T(\epsilon^T)\}_{\epsilon^T}\right).$$

The *lifetime utility function* U increases in all its arguments. The notation $\{c_t(\epsilon^t)\}_{\epsilon^t}$ indicates that consumption at date t takes multiple values, depending on history. For example, in the environment of figure 1.1, we have $\{c_1(\epsilon^1)\}_{\epsilon^1} = \{c_1(\text{up}), c_1(\text{down})\}$.

Let $c \equiv (c_0, \{c_1(\epsilon^1)\}_{\epsilon^1}, \ldots, \{c_T(\epsilon^T)\}_{\epsilon^T})$ denote a stochastic consumption path and let U be differentiable, strictly increasing, and concave. The *marginal rate of substitution* between $c_t(\epsilon^t)$ and $c_s(\epsilon^s)$ equals

$$\frac{\partial U(c)/\partial c_t(\epsilon^t)}{\partial U(c)/\partial c_s(\epsilon^s)}.$$

Function U is *homothetic* if marginal rates of substitution do not change when c is scaled up or down. With homothetic preferences, relative demands are invariant to shifts of the budget line that is, homothetic preferences generate linear *Engel curves*. Suppose that $U(c) = g(h(c))$ where g is strictly increasing and h is homogeneous of degree $n \geq 1$. By Euler's homogeneous function theorem the partial derivatives $\partial h(c)/\partial c_t(\epsilon^t)$ and $\partial h(c)/\partial c_s(\epsilon^s)$ then are homogeneous of degree $n-1$. This implies that U is homothetic.

We often assume that preferences are *additively separable* across time and histories that is, U is a weighted sum. The weight attached to date t equals β^t where $\beta \in [0, 1)$

denotes the psychological *discount factor* which measures the degree of patience. The weight attached to a particular history equals the probability that this history occurs. A consumption sequence thus is evaluated according to the discounted *expected utility* that it generates,

$$U = \mathbb{E}_0 \left[\sum_{t=0}^{T} \beta^t u(c_t(\epsilon^t)) \right].$$

Here, $\mathbb{E}_s$ denotes the mathematical expectation conditional on information available at date s, history ϵ^s. Separability across time and histories implies that the marginal utility of consumption at a date and history is independent of consumption at other dates and histories. This property often simplifies the analysis.

The *period utility function* or *felicity function* u exhibits strictly positive, decreasing marginal utility. Unless otherwise noted, we assume that it is continuously differentiable; marginal utility is strictly decreasing; and consumption at each date is essential, $\lim_{c\downarrow 0} u'(c) = \infty$.

We sometimes restrict u to be of the *constant intertemporal elasticity of substitution* (CIES) or equivalently, *constant relative risk aversion* form. Function u then is given by

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma} \text{ for } \sigma > 0,\ \sigma \neq 1.$$

This functional form is not defined for $\sigma = 1$, but applying L'Hôpital's rule implies $\lim_{\sigma\to 1} u(c) = \ln(c)$; the logarithmic function thus constitutes a limiting case.

As the name suggests, CIES preferences exhibit constant relative risk aversion and a constant intertemporal elasticity of substitution. To see the former, recall that the *coefficient of relative risk aversion* is defined as $-u''(c)c/u'(c)$; with CIES preferences this reduces to σ. To see the latter, recall that the *elasticity of substitution* measures how strongly a change of relative price affects relative demand. With CIES preferences, the elasticity of the ratio c_{t+1}/c_t with respect to the relative price of c_{t+1} and c_t reduces to $1/\sigma$. As we will see in chapter 2, CIES preferences simplify the equilibrium conditions in dynamic models.

Figure 1.2 illustrates the role of the elasticity of substitution. The top row of the figure plots $u(c)$ and the bottom row plots indifference curves of the (homothetic) utility function $U = u(c_0) + \beta u(c_1)$. As σ increases the indifference curves gain curvature; variations in c_1/c_0 therefore are associated with larger changes in the marginal rate of substitution.

1.2.3 Technology

Firms employ a *production function*, f, that maps inputs of physical capital, K, and labor, L, into output. Unless otherwise noted, we assume that the production function is *neoclassical*. That is, f exhibits strictly positive and diminishing marginal products as well as *constant returns to scale*,

$$f_K(K,L), f_L(K,L) > 0;\;\; f_{KK}(K,L), f_{LL}(K,L) < 0;\;\; \phi f(K,L) = f(\phi K, \phi L),\ \phi > 0.$$

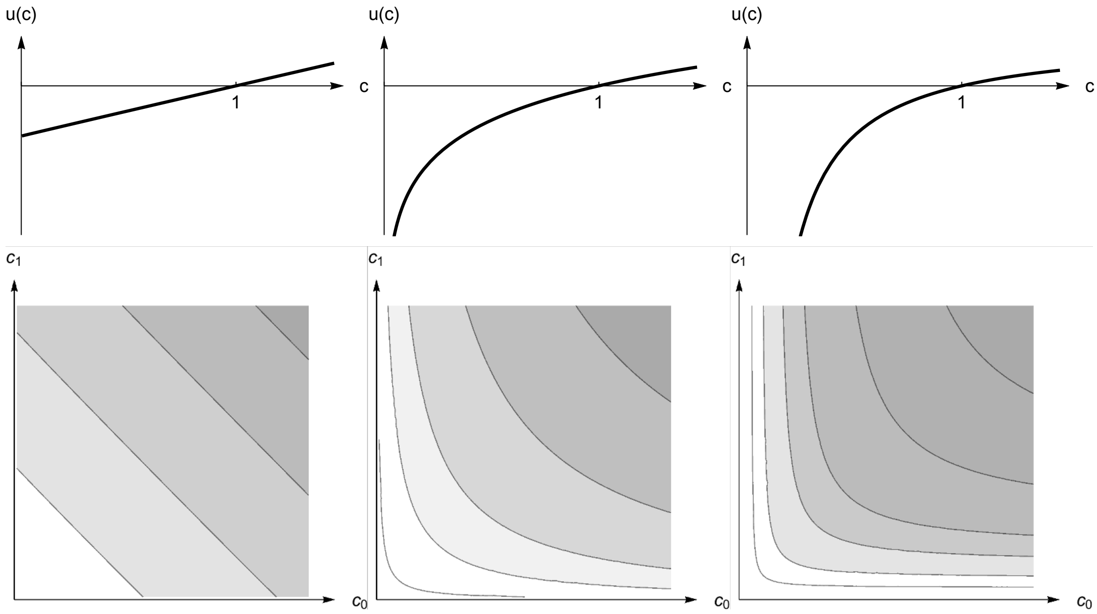

Figure 1.2: CIES preferences: Period utility function and indifference curves for very small, intermediate, and large σ (from left to right).

Here, subscripts denote partial derivatives, $f_K(K,L) \equiv \partial f(K,L)/\partial K$ and $f_{KK}(K,L) \equiv \partial^2 f(K,L)/(\partial K)^2$; we use this notation throughout the book.

Due to constant returns to scale, output coincides with total factor payments to the suppliers of K and L if the rental rates of K and L equal the respective marginal products. On competitive factor markets this is the case. Also due to constant returns to scale, output per worker as well as marginal products only depend on the capital-labor ratio, $k \equiv K/L$, not on K and L individually. Both these facts follow from Euler's homogeneous function theorem.

The production function satisfies the *Inada conditions* when

$$\lim_{K\downarrow 0} f_K(K,L) = \lim_{L\downarrow 0} f_L(K,L) = \infty, \ \lim_{K\to\infty} f_K(K,L) = \lim_{L\to\infty} f_L(K,L) = 0.$$

The Inada conditions help guarantee interior equilibria.

The *constant elasticity of substitution* (CES) production function,

$$f(K,L) = \left(\alpha K^{1-\frac{1}{\theta}} + (1-\alpha)L^{1-\frac{1}{\theta}}\right)^{\frac{\theta}{\theta-1}}, \ \theta > 0, \ \alpha \in (0,1),$$

constitutes a tractable example of a neoclassical production function. The elasticity of substitution between K and L is constant in this case and equals θ. For $\theta \to \infty$, the CES production function approaches a linear production function, and for $\theta \to 0$ it approaches the Leontief production function.

For $\theta \to 1$, the CES production function converges to the *Cobb-Douglas production function*,

$$f(K,L) = K^\alpha L^{1-\alpha}.$$

When production factors are paid their marginal products the Cobb-Douglas production function implies constant factor shares, $Kf_K(K, L)/f(K, L) = \alpha$ and $Lf_L(K, L)/f(K, L) = 1 - \alpha$.

1.3 Bibliographic Notes

The Walrasian equilibrium notion is due to Walras (1874) and other representatives of the (pre)marginalist school (von Thünen, Cournot, Dupuis, Gossen, Jevons, Menger). Arrow and Debreu (1954) and McKenzie (1954) use fixed point arguments to prove existence of general equilibrium and Debreu (1959) proves the welfare theorems. Bewley (1972) proves that in economies with a finite number of *infinitely* lived, impatient households an equilibrium still exists and is Pareto optimal.

Arrow (1953; 1964) and Debreu (1959, 7) define commodities with reference to the event tree. Cobb and Douglas (1928) discuss the production function named after them.

Related Topics The notion of macroeconomic *equilibrium* has evolved since Walras (1874). Hicks (1939) envisions interdependent markets that operate sequentially and he distinguishes between *temporary equilbrium* in spot markets (conditional on expectations about the future), and *equilibrium over time* when expected and actual prices coincide. Stigum (1969) relaxes the restriction on expectations and Grandmont (1977) discusses the notion of a *sequence of temporary equilibria*, possibly including learning. Common price expectations across agents and consistent plans based on these expectations form an *equilibrium of plans, prices, and price expectations* (Hahn, 1971; Radner, 1972). When agents have heterogeneous information sets market prices may reveal some of the unobserved fundamentals and agents may use a model of the relationship between equilibrium prices and fundamentals to infer the latter; in a *rational expectations equilibrium* the prediction model used by the agents in the model is the correct model (Lucas, 1972). Barro and Grossman (1971) study a general *disequilibrium* model where excess supplies and demands in different markets affect each other.

More broadly, the methodological approach to studying *macroeconomic* questions has changed profoundly over the last 60 years. In the late 1950s, microeconomic reasoning coexists with Keynesian arguments (Keynes, 1936; Hicks, 1939) which often rest on weaker choice theoretic foundations. During the 1960s, empirical macroeconomics suffers setbacks, not least because the *Phillips curve* relationship between observed unemployment and (wage) inflation proves less stable than expected. Friedman's (1968, p. 8) dictum that the Phillips curve presumes "a world in which everyone anticipated that nominal prices would be stable ... whatever happened" stimulates the search for models that better reconcile micro- and macroeconomics (Phelps, 1970). Building on Muth (1961), Sargent (1971) and Lucas (1972) promote the rational expectations consistency requirement.

By the late 1970s, the profession looses faith in the *neoclassical synthesis* (Samuelson, 1955) of Keynesian and neoclassical economics and in particular, in policy analysis based on large-scale macroeconometric models. It becomes clear that optimizing behavior also concerns expectation formation; reduced form relationships in models without micro foundations are not policy-invariant (Lucas, 1976); and "claims for identification in [large-scale statistical macroeconomic] models" are unfounded (Sims, 1980, p. 1). Early micro founded dynamic general equilibrium models in the 1980s lack plausible frictions and abstract from heterogeneity which limits their relevance for applied purposes. Modern macroeconomic models for applied purposes—models of *dynamic stochastic general equilibrium*—feature heterogeneity, time, risk, and diverse frictions.

Mas-Colell, Whinston and Green (1995) cover microeconomic theory and Eatwell, Milgate and Newman (1989) provide an overview over topics in general equilibrium theory. Niehans (1994) covers the history of economic thought until 1980.

Chapter 2

Consumption and Saving

The consumption-saving tradeoff of households constitutes the backbone of most modern macroeconomic models. In this chapter, we study the household's dynamic utility maximization problem and the induced demand functions. Throughout, we abstract from risk and assume that leisure does not enter preferences.

2.1 Consumption Smoothing

Consider a household that owns a (possibly negative) stock of assets, a_t. The household receives (or pays) interest income on the assets, $a_t(R_t - 1)$, where R_t denotes the gross interest rate, and receives exogenous wage income, w_t. The stock of assets and the two incomes fund consumption, c_t, or can be carried into the next period, a_{t+1}. The *dynamic budget constraint*

$$a_{t+1} = a_t R_t + w_t - c_t$$

represents the resulting asset dynamics. Rearranging terms, the dynamic budget constraint states that the change in the asset position, $a_{t+1} - a_t$, equals *saving* or income minus consumption.

2.1.1 Two Periods

With two periods, the household's objective function is given by

$$u(c_0) + \beta u(c_1).$$

When the household has no assets to start with ($a_0 = 0$), the dynamic budget constraints at the two dates read

$$\begin{aligned} a_1 &= w_0 - c_0, \\ a_2 &= a_1 R_1 + w_1 - c_1. \end{aligned}$$

Since the household dies at the end of date $t = 1$ nobody will lend it resources at that date; terminal household assets therefore must be non-negative, $a_2 \geq 0$. Moreover, carrying strictly positive assets into date $t = 2$ would be wasteful because the household cannot consume after its death. The optimal saving choice at date $t = 1$ thus is $a_2 = 0$.

Using this result and combining the two dynamic budget constraints, we arrive at the *intertemporal budget constraint*

$$c_0 + \frac{c_1}{R_1} = w_0 + \frac{w_1}{R_1}.$$

The terms on the left-hand side represent total spending on the two goods, consumption in the first and second period, c_0 and c_1 respectively. Consumption in the first period is the *numeraire*, its price is normalized to unity. The relative price of consumption in the second period is given by the inverse of the gross interest rate, $1/R_1$. Intuitively, reducing consumption in the first period by one unit raises saving and increases consumption in the second period by R_1 units. One unit of first-period consumption therefore buys R_1 units of second-period consumption, or one unit of second-period consumption costs $1/R_1$ units of first-period consumption.

The terms on the right-hand side of the intertemporal budget constraint represent the household's wealth, that is the date $t = 0$ market value of first- and second-period wage income. Note that the intertemporal budget constraint is isomorphic to the budget constraint in a static model of consumer choice.

To find the household's optimal level of saving in the first period, we may solve the dynamic budget constraints for consumption and substitute the resulting expressions into the objective function. The household's program then reads[1]

$$\max_{a_1} u(w_0 - a_1) + \beta u(a_1 R_1 + w_1).$$

An interior solution to this program satisfies the first-order condition or *Euler equation*

$$u'(c_0) = \beta R_1 u'(c_1) \text{ or } \frac{u'(c_0)}{\beta u'(c_1)} = R_1$$

where we re-introduce the variables c_0 and c_1 for ease of notation.

The second representation of the Euler equation states that the marginal rate of substitution between current and future consumption is equated with the relative price between the two goods. This is the same condition as in a static model with apples and oranges, say, where the price line is tangent to the highest indifference curve.

The Euler equation characterizes optimal second-period consumption relative to first-period consumption and thus, the slope of the optimal consumption path but not its level. Three factors determine whether and how strongly consumption increases or decreases over time. First, β. More patience increases the weight given to future utility and thus, the slope of the optimal consumption path. Formally, a higher β implies a higher ratio $u'(c_0)/u'(c_1)$ and thus (if u is strictly concave), a higher c_1/c_0. Second, R_1. A higher interest rate renders second-period consumption cheaper, also implying

[1]Throughout, we abstract from non-negativity constraints on consumption unless they may be relevant.

a higher c_1/c_0. Third, the curvature of the marginal utility function (recall figure 1.2). It determines how strongly a change of β or R_1 translates into a steeper or flatter consumption profile. More curvature implies a stronger *consumption smoothing* motive that is, less willingness to intertemporally substitute.

To solve for the equilibrium consumption levels in terms of the exogenous variables we combine the Euler equation and the intertemporal budget constraint (or the two dynamic budget constraints). If the period utility function is of the CIES type (such that the Euler equation reads $c_0^{-\sigma} = \beta R_1 c_1^{-\sigma}$) this yields

$$c_0 = \left(w_0 + \frac{w_1}{R_1}\right) \Bigg/ \left(1 + \frac{(\beta R_1)^{1/\sigma}}{R_1}\right).$$

From the budget constraint, we may also solve for a_1 and c_1.

We have completely characterized optimal consumption conditional on β, u, R_1, w_0, and w_1. Two important results emerge. First, optimal consumption depends on wealth, $w_0 + w_1/R_1$, or *permanent income*, not only on contemporaneous income as with a Keynesian consumption function. This is a consequence of the household's desire to smooth consumption over the life cycle if marginal utility is decreasing ($u'' < 0$), and its ability to do so by means of saving or borrowing.

Second, a change of interest rate affects optimal consumption threefold, through two *income* or *wealth effects* and one *substitution effect*. First, if $w_1 > 0$, an increase in the interest rate reduces wealth because it lowers the date $t = 0$ market value of future labor income. This leads the household to consume less in both periods. Second, for given quantities c_0 and $c_1 > 0$, an increase in the interest rate lowers the cost of the bundle (c_0, c_1) expressed in terms of the numeraire. The higher purchasing power leads the household to consume more of both goods. Finally, from the Euler equation, it is optimal to substitute towards the cheaper good. An increase in the interest rate thus leads the household to increase c_1 relative to c_0. The strength of the substitution effect depends on the intertemporal elasticity of substitution, $1/\sigma$.

Figure 2.1 illustrates the three effects. Point A depicts the equilibrium at a low interest rate; it is the tangency point of the budget line and the highest indifference curve that the household may attain. Point D depicts the equilibrium at a higher interest rate, represented by a steeper budget line. The substitution effect corresponds to the distance between points A and B; the income effect due to increased purchasing power to the distance between points B and C; and the wealth effect due to lower discounted second-period labor income to the distance between points C and D. The figure is plotted for the logarithmic utility case, $\sigma = 1$, such that the income and substitution effects on c_0 exactly cancel.

2.1.2 More Periods

More generally, the household's program comprises a multi-period objective function and multiple dynamic budget constraints in addition to the initial and terminal condi-

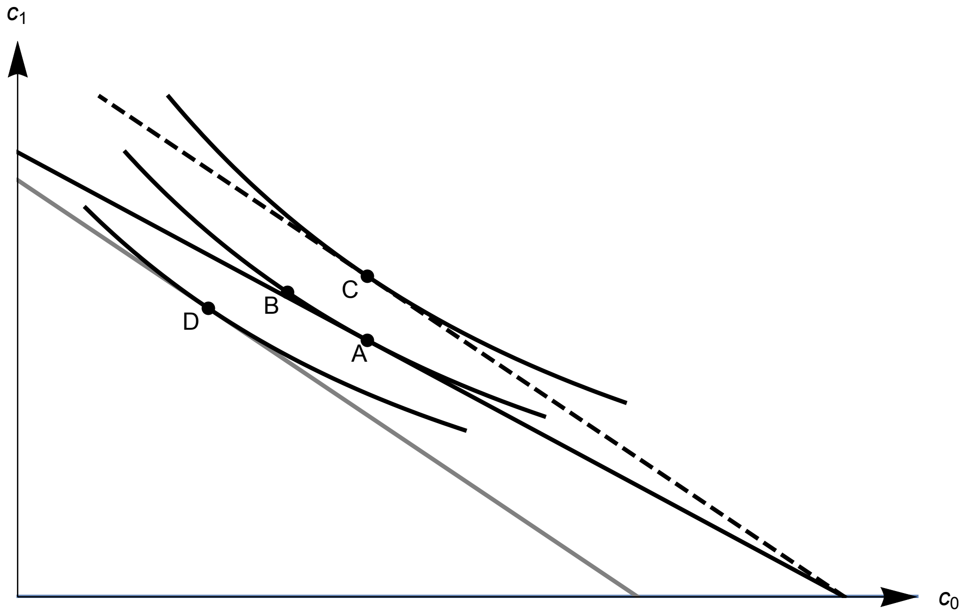

Figure 2.1: Effects of an interest rate change: Income/wealth and substitution effects.

tions:

$$\max_{c_0,\ldots,c_T,a_1,\ldots,a_{T+1}} \sum_{t=0}^{T} \beta^t u(c_t) \text{ s.t. } a_{t+1} = a_t R_t + w_t - c_t,\ a_0 R_0 \text{ given},\ a_{T+1} \geq 0.$$

In parallel with the strategy adopted in the two-period case, we can confront this problem by solving each of the dynamic budget constraints for consumption and substituting the resulting expressions into the objective function. Conjecturing again that a_{T+1} optimally equals zero this yields

$$\max_{a_1,\ldots,a_T} \sum_{t=0}^{T} \beta^t u(a_t R_t + w_t - a_{t+1}) \text{ s.t. } a_0 R_0 \text{ given},\ a_{T+1} = 0.$$

Differentiating with respect to the choice variables, we find an Euler equation for each date t.

Lagrangian

An alternative strategy uses the intertemporal budget constraint. The latter can be derived, as before, by combining the dynamic budget constraints:

$$\begin{aligned} a_{T+1} &= a_T R_T + w_T - c_T \\ &= (a_{T-1} R_{T-1} + w_{T-1} - c_{T-1}) R_T + w_T - c_T \\ &= \ldots \\ &= a_0 R_0 R_1 \cdots R_T + (w_0 - c_0) R_1 R_2 \cdots R_T + (w_1 - c_1) R_2 \cdots R_T \\ &\quad + \ldots + (w_{T-1} - c_{T-1}) R_T + (w_T - c_T). \end{aligned}$$

Let $q_t \equiv (R_1 R_2 \cdots R_t)^{-1}$ denote the price of date-t consumption at the initial date and define $q_0 \equiv 1$. Multiplying the intertemporal budget constraint by q_T and using the optimality condition $a_{T+1} = 0$ yields

$$q_T a_{T+1} = 0 = a_0 R_0 + \sum_{t=0}^{T} q_t (w_t - c_t) \text{ or } \sum_{t=0}^{T} q_t c_t = a_0 R_0 + \sum_{t=0}^{T} q_t w_t.$$

In parallel to the two-period case, the intertemporal budget constraint equalizes lifetime consumption spending and lifetime wealth at market prices.

Replacing the $T + 1$ dynamic budget constraints with the single intertemporal budget constraint, the household's program can be expressed as

$$\max_{c_0,\ldots,c_T} \sum_{t=0}^{T} \beta^t u(c_t) \text{ s.t. } a_0 R_0 + \sum_{t=0}^{T} q_t (w_t - c_t) = 0.$$

Forming the Lagrangian (see appendix A.1)

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t u(c_t) + \lambda [a_0 R_0 + \sum_{t=0}^{T} q_t (w_t - c_t)]$$

and differentiating yields T first-order conditions,

$$\beta^t u'(c_t) = \lambda q_t.$$

They state that marginal utility from consumption in a period equals the price of consumption in that period, q_t, times the *multiplier* attached to the intertemporal budget constraint, λ. Since the multiplier measures the effect of a marginal relaxation of the constraint on the maximized objective function, λ represents the *shadow value of wealth*. Combining the first-order conditions at date t and $t + 1$ yields the Euler equation, $u'(c_t) = \beta R_{t+1} u'(c_{t+1})$. To solve for the optimal consumption levels, we combine the Euler equations and the intertemporal budget constraint.

Yet another approach to solving the household's program relies on forming a Lagrangian that incorporates the dynamic budget constraints and the terminal condition $a_{T+1} \geq 0$, and deriving the optimality condition $a_{T+1} = 0$ rather than imposing it from the beginning. This Lagrangian reads (see appendix A.1)

$$\mathcal{L} = \sum_{t=0}^{T} \{\beta^t u(c_t) - \lambda_t [a_{t+1} - (a_t R_t + w_t - c_t)]\} + \mu a_{T+1}.$$

The first-order conditions with respect to c_t and a_{t+1} are given by

$$\beta^t u'(c_t) = \lambda_t,$$
$$\lambda_t = \lambda_{t+1} R_{t+1},\ t = 0, \ldots, T - 1,$$

respectively. The first-order condition with respect to a_{T+1} is $\lambda_T = \mu$ and the complementary slackness condition is given by $\mu a_{T+1} = 0$.

Combining the first-order conditions with respect to consumption again yields the Euler equation, $u'(c_t) = \beta R_{t+1} u'(c_{t+1})$. Moreover, non-satiation ($u' > 0$) implies $\lambda_T > 0$ and thus $\mu > 0$ which in turn implies $a_{T+1} = 0$, the *transversality condition* we had informally argued before.

Dynamic Programming

We may also solve the household's program using *dynamic programming* techniques. Recall that the *indirect utility function* gives the maximal utility as a function of the parameter(s) of the utility maximization problem. In the context of a consumption saving program, the parameters of the indirect utility function include the level of initial assets, a_t, as well as preference parameters, wages and interest rates over the remaining lifetime.

The household's *value function* is the indirect utility function and the corresponding parameters are referred to as the *state* or *state variable(s)* which summarize both the effects of past decisions and current information about the future. Incorporating all exogenous elements of the state into the time subscript, we can express the value function at date t as a function V_t of assets. (Alternatively, we could express it as a function of assets and the time horizon that is left, that is as a function V_{T-t} of assets.) Note that in contrast to wages and interest rates, a_t is an endogenous state variable: It constitutes a parameter in the program at date t but is a choice variable in earlier periods.

For brevity, let DBC_t denote the dynamic budget constraint at date t and let $\mathcal{C}$ denote the set of dynamic budget constraints at date $t+1$ and later as well as the terminal condition $a_{T+1} \geq 0$. The value function at date t then satisfies

$$V_t(a_t) = \max_{\{c_s, a_{s+1}\}_{s=t}^T} \sum_{s=t}^T \beta^{s-t} u(c_s) \;\text{ s.t. }\; \text{DBC}_t, \mathcal{C}, a_t \text{ given.}$$

Alternatively, it can be represented recursively as

$$V_t(a_t) = \max_{c_t, a_{t+1}} \{u(c_t) + \beta V_{t+1}(a_{t+1})\} \;\text{ s.t. }\; \text{DBC}_t, a_t \text{ given.}$$

To see this, simply rearrange terms exploiting the additive separability of preferences:

$$\begin{aligned}
V_t(a_t) &= \max_{\{c_s, a_{s+1}\}_{s=t}^T} \sum_{s=t}^T \beta^{s-t} u(c_s) \;\text{ s.t. }\; \text{DBC}_t, \mathcal{C}, a_t \text{ given} \\
&= \max_{c_t, a_{t+1}} u(c_t) + \left(\max_{\{c_s, a_{s+1}\}_{s=t+1}^T} \sum_{s=t+1}^T \beta^{s-t} u(c_s) \;\text{ s.t. }\; \mathcal{C}, a_{t+1} \text{ given} \right) \;\text{ s.t. }\; \text{DBC}_t, a_t \text{ given} \\
&= \max_{c_t, a_{t+1}} u(c_t) + \beta \left(\max_{\{c_s, a_{s+1}\}_{s=t+1}^T} \sum_{s=t+1}^T \beta^{s-(t+1)} u(c_s) \;\text{ s.t. }\; \mathcal{C}, a_{t+1} \text{ given} \right) \;\text{ s.t. }\; \text{DBC}_t, a_t \text{ given} \\
&= \max_{c_t, a_{t+1}} u(c_t) + \beta V_{t+1}(a_{t+1}) \;\text{ s.t. }\; \text{DBC}_t, a_t \text{ given.}
\end{aligned}$$

We are confronted with a system of functional equations often referred to as *Bellman equations*. These *functional equations* stipulate equality of functions (rather than functions evaluated at specific points). Substituting the dynamic budget constraint yields a compact representation of the Bellman equation,

$$V_t(a_t) = \max_{a_{t+1}} u(a_t R_t + w_t - a_{t+1}) + \beta V_{t+1}(a_{t+1}),$$

which has to hold at all dates and for all feasible values of a_t.

Since T is finite we can solve the system of Bellman equations by backward induction. To start the induction, note that $V_{T+1}(a_{T+1}) = 0$ for all a_{T+1}. At date T, this implies the optimal choice $a_{T+1} = 0$ and thus, the value function $V_T(a_T) = u(a_T R_T + w_T)$. Using this result and the Bellman equation at date $T-1$ we can characterize the optimal choice and the value function at date $T-1$. Proceeding backward, we can solve for all value functions V_t and *policy functions* g_t say; the latter give the optimal value of the choice variable as a function of the state, $a_{t+1} = g_t(a_t)$.[2]

To derive the Euler equation, we do not need to know the functional form of the value functions. It suffices to use the first-order and envelope conditions,

$$\begin{aligned} u'(c_t) &= \beta V'_{t+1}(a_{t+1}), \\ V'_t(a_t) &= u'(c_t)(R_t - g'_t(a_t)) + \beta V'_{t+1}(a_{t+1}) g'_t(a_t) \\ &= u'(c_t) R_t. \end{aligned}$$

The first-order condition in the first line results from differentiating the right-hand side of the Bellman equation with respect to the choice variable. An optimal choice of a_{t+1} assures that this condition is satisfied. The *envelope condition* in the second line results from differentiating the Bellman equation with respect to the state variable. The last equality follows from substituting the first-order condition into the second condition—this is the envelope theorem at work (see below). Combining the first-order condition and the envelope condition (evaluated at $t+1$) yields the Euler equation, $u'(c_t) = \beta R_{t+1} u'(c_{t+1})$. It can also be expressed as the functional equation

$$u'(a_t R_t + w_t - g_t(a_t)) = \beta R_{t+1} u'(g_t(a_t) R_{t+1} + w_{t+1} - g_{t+1}(g_t(a_t)))$$

which must hold for all feasible values of a_t.

Figure 2.2 illustrates the envelope condition at date $T-1$. The figure plots the level curves of the right-hand side of the Bellman equation at date $T-1$, $\text{RHS}(a_T; a_{T-1}) \equiv u(a_{T-1} R_{T-1} + w_{T-1} - a_T) + \beta u(a_T R_T + w_T)$, against a_{T-1} (the state variable) and a_T (the choice variable). Darker areas indicate higher values of RHS. For a given value of the state, for instance the horizontal coordinate of point A, the maximand is a concave function of the choice variable that reaches a maximum at the optimal choice, for instance the vertical coordinate of point A. A small increase in the state, indicated by the solid arrow, has two effects on RHS. A direct one; and an indirect one, due to the adjustment of the optimal choice indicated by the dashed arrow to point B. Since the level curve at point A is vertical the level curve through the tip of the solid arrow is vertical as well, to the first order. Accordingly, the indirect utility effect is of second order: The only first-order effect of an infinitesimal change of the state variable on the value function is the direct one.

[2]It is straightforward to write a computer program that iteratively computes approximations of V_t and g_t for $t = T, T-1, T-2, \ldots$.

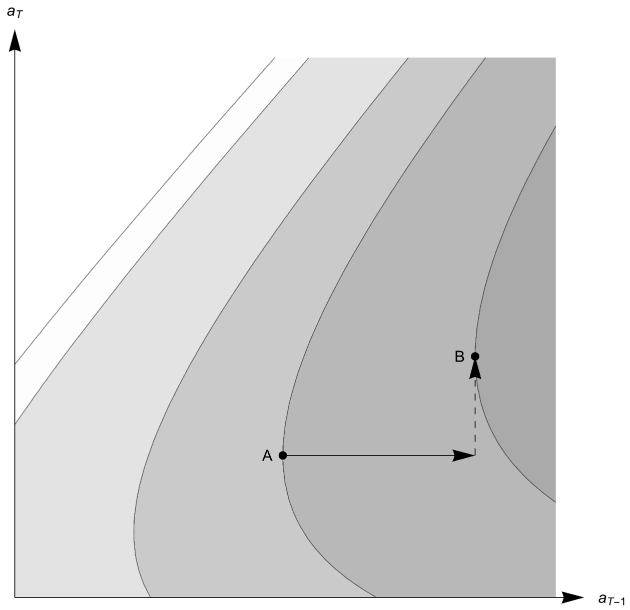

Figure 2.2: Envelope condition: Level curves of the right-hand side of the Bellman equation.

2.1.3 Infinite Horizon

There are several reasons to consider optimization of households (and other agents) over an *infinite horizon*, $T \to \infty$. First, because this can be interpreted as reflecting intergenerational altruism: Parents care about the utility of their children who in turn care about the utility of their children, and so on. Second, because an infinite horizon can be interpreted as reflecting a time invariant survival probability. And third, because eliminating time as a state variable makes the program simpler.

To derive the household's intertemporal budget constraint in the infinite-horizon case, we need to specify an appropriate terminal condition. This is given by the *no-Ponzi-game condition* $\lim_{T\to\infty} q_T a_{T+1} \geq 0$. Note that the no-Ponzi-game condition generalizes the constraint $q_T a_{T+1} \geq 0$ from the finite horizon case, not its reduced form $a_{T+1} \geq 0$. In fact, a generalization of the latter, to $\lim_{T\to\infty} a_{T+1} \geq 0$ say, would constitute an unnecessarily tight constraint that prevents the household from holding any debt in the long run.

In contrast, the no-Ponzi-game condition $\lim_{T\to\infty} q_T a_{T+1} \geq 0$ only rules out debt positions in the long run that grow at a rate weakly higher than the interest rate. The household thus is prevented from permanently rolling over debt, including interest, and never servicing it. While the no-Ponzi-game condition guarantees that going forward, the present discounted value of debt service fully covers the outstanding debt it does not impose restrictions on the time profile of the debt service. For example, the household may once and for all pay back all outstanding debt, or it may never repay the principal but instead pay interest on the liability forever after.

Using the no-Ponzi-game condition and following the same steps as in the finite-horizon case, we can derive the infinite-horizon intertemporal budget constraint

$$a_0 R_0 + \lim_{T\to\infty} \sum_{t=0}^{T} q_t(w_t - c_t) = \lim_{T\to\infty} q_T a_{T+1} \geq 0.$$

By the same logic as in the finite-horizon case, optimality requires setting $\lim_{T\to\infty} q_T a_{T+1}$ as small as possible (see appendix B.1). The intertemporal budget constraint therefore reduces to $a_0 R_0 + \sum_{t=0}^{\infty} q_t(w_t - c_t) = 0$ and the household's program can be represented as

$$\max_{\{c_t\}_{t\geq 0}} \sum_{t=0}^{\infty} \beta^t u(c_t) \text{ s.t. } a_0 R_0 + \sum_{t=0}^{\infty} q_t(w_t - c_t) = 0.$$

Forming the Lagrangian

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \lambda[a_0 R_0 + \sum_{t=0}^{\infty} q_t(w_t - c_t)]$$

and differentiating yields the same first-order conditions as before,

$$\beta^t u'(c_t) = \lambda q_t,$$

and thus, the Euler equation.

Turn next to dynamic programming. With an infinite horizon, the household's horizon always is the same, independently of how many periods have gone by. The structure of the optimization problem therefore is independent of time—the problem is *time autonomous*—unless wages, interest rates, or preferences are time dependent. In the time autonomous case, the Bellman equation can be expressed as

$$V(a_\circ) = \max_{a_+} u(a_\circ R + w - a_+) + \beta V(a_+).$$

Note that the value functions on the left- and right-hand side are identical (the functions do not have time subscripts), in contrast with the finite-horizon case. The state variable $a_\circ$ and the choice variable a_+ are written without time subscripts to indicate the time autonomous nature of the program.

Although in the infinite-horizon case no final period exists, one can nevertheless find the value function V of the infinite-horizon problem by means of an iterative procedure that parallels the solution strategy in the finite-horizon case. This follows from mathematical results which establish that under certain conditions, (i) the value function V solving the time-autonomous Bellman equation is unique, and (ii) starting from any value function guess (for example the function that started the recursion in the finite-horizon case, $V_{T+1}(a_{T+1}) = 0$), the iterative procedure applied in the finite-horizon case yields a sequence of value functions that converges to V (see appendix A.2). When working with a computer, an approximation of the infinite-horizon value function thus can be found by running exactly the same code as in the finite horizon case except that the iterative procedure only is stopped when the sequence of value function approximations has converged.

2.2 Extensions

2.2.1 Borrowing Constraint

We have assumed that households may freely borrow as long as they satisfy the intertemporal budget constraint that is, as long as they are solvent. But borrowing against future wage income may be difficult, for example because a potential lender does not have sufficient information about the future income stream or cannot enforce repayment. This renders future wage income *illiquid* and gives rise to a new constraint—a *liquidity* or *borrowing constraint*—in addition to the intertemporal budget constraint.

The simplest possible borrowing constraint excludes all borrowing against future wage income. The household's financial assets then must be positive at all times, $a_{t+1} \geq 0$. A binding borrowing constraint is costly because it prevents consumption smoothing. To see this, consider a two-period model with strictly concave preferences and suppose that absent a borrowing constraint, optimal consumption in the first period exceeds liquid wealth, $w_0 + a_0 R_0$; implementing the optimal consumption plan thus requires borrowing, $a_1 < 0$. The borrowing constraint renders this infeasible. Constrained optimal consumption then equals $(c_0, c_1) = (w_0 + a_0 R_0, w_1)$ that is, consumption follows income, as with a Keynesian consumption function. Note that we have identified a new saving motive: reduced borrowing (that is, increased saving) due to a binding borrowing constraint.

More formally, the Lagrangian associated with the constrained saving problem reads

$$\mathcal{L} = u(c_0) + \beta u(c_1) - \lambda \left(c_0 + \frac{c_1}{R_1} - w_0 - \frac{w_1}{R_1} - a_0 R_0 \right) + \mu(w_0 + a_0 R_0 - c_0),$$

where the non-negative multiplier μ represents the shadow cost of the borrowing constraint $w_0 + a_0 R_0 - c_0 \geq 0$. Differentiating with respect to c_0 and c_1 and combining the two conditions yields the modified Euler equation

$$u'(c_0) = \beta R_1 u'(c_1) + \mu.$$

A binding borrowing constraint, $\mu > 0$, increases the slope of the equilibrium consumption path. Moreover, $\mu > 0$ and the complementary slackness condition, $\mu(w_0 + a_0 R_0 - c_0) = 0$, imply $c_0 = w_0 + a_0 R_0$, in line with the heuristic argument above.

2.2.2 Non-Geometric Discounting and Time Consistency

Until now, we have posited that the sequence of psychological discount factors is geometrically declining, $1, \beta, \beta^2, \beta^3, \ldots$. Under this assumption, a household that reoptimizes period by period opts to continue with the consumption plan chosen earlier in time. That is, if the household optimally chose the consumption plan $(c_t, \overrightarrow{c_{t+1}})$ with $\overrightarrow{c_{t+1}} \equiv \{c_{t+1}, c_{t+2}, \ldots\}$ at date t, then pursuing the plan $\overrightarrow{c_{t+1}}$ once time has progressed to date $t+1$ remains optimal. As a consequence, it does not matter whether we assume that the household chooses the consumption plan at date $t = 0$ once and for all, under *commitment*, or whether it re-optimizes period by period. In this sense, the initial consumption plan is *time consistent*.

Under more general assumptions about the psychological discount factor sequence the consumption plan at date $t = 0$ need not be time consistent. Two households, one re-optimizing period by period and the other acting under commitment, may end up with different consumption paths although their preferences at date $t = 0$ and their budget sets are identical.

Consider an extreme example in a three-period model where the household discounts all future utility at factor β. At date $t = 0$, the household has preferences $u(c_0) + \beta(u(c_1) + u(c_2))$ while at date $t = 1$, preferences are given by $u(c_1) + \beta u(c_2)$. For simplicity, let $R_t = 1, w_t = w$. The optimal consumption path as of date $t = 0$ then solves the problem

$$\max_{c_0,c_1,c_2} u(c_0) + \beta(u(c_1) + u(c_2)) \text{ s.t. } c_0 + c_1 + c_2 = 3w + a_0,$$

which yields $c_1 = c_2$ (assuming $u'' < 0$). A household who can commit implements this solution.

In contrast, for given a_1 the optimal consumption path as of date $t = 1$ solves

$$\max_{c_1,c_2} u(c_1) + \beta u(c_2) \text{ s.t. } c_1 + c_2 = 2w + a_1,$$

which yields $c_1 > c_2$. Absent commitment, a household re-optimizing at date $t = 1$ thus does not implement the path that is optimal from the perspective of date $t = 0$. Without commitment, the ex-ante optimal consumption plan cannot be implemented because it is *time inconsistent*.

Without commitment, the two selves of the household play a game against each other. The first self chooses c_0 and a_1. The second self chooses c_1, a_2 and c_2 conditional on a_1. Since the second self chooses $c_1 > c_2$, the first self cannot implement the ex-ante optimal plan. Anticipating the second self's choice, the first self solves

$$\max_{c_0,c_1,c_2} u(c_0) + \beta(u(c_1) + u(c_2)) \text{ s.t. } c_0 + c_1 + c_2 = 3w + a_0, u'(c_1) = \beta u'(c_2)$$

where the second constraint, the Euler equation of the second self, reflects the consumption choice from date $t = 1$ onward. By choosing a_1 the first self affects the state variable at date $t = 1$ and may thus influence the action taken by the second self.

Instruments that serve as additional state variables can be helpful for the first self even if these instruments would be irrelevant under commitment. Ulysses, for example, manages to bypass the island of the Sirens by having his crew put wax into their ears (so that they cannot hear the Sirens sing) and himself tied to the mast (so that he cannot change the course of the vessel).

2.2.3 Multiple Goods

Consider a two-period lived household that consumes two goods in each period. Their quantities, d_t and e_t, are aggregated into a CES consumption index,

$$c_t(d_t, e_t) = \left(\delta^{\frac{1}{\theta}} d_t^{1-\frac{1}{\theta}} + \varepsilon^{\frac{1}{\theta}} e_t^{1-\frac{1}{\theta}}\right)^{\frac{\theta}{\theta-1}}, \ \delta + \varepsilon = 1, \ \theta > 0,$$

with elasticity of substitution equal to θ. The price of d_t is normalized to unity and the relative price of e_t is denoted p_t. Intertemporal preferences are described by the utility function $u(c_0) + \beta u(c_1)$.

The household's consumption choice has an *intratemporal* dimension (the trade-off between d_t and e_t) and an *intertemporal* one (the trade-off between c_0 and c_1). Focusing first on the intratemporal trade-off, consider the problem of maximizing c_t subject to a given amount of spending, $z_t = d_t + p_t e_t$. The solution to this problem is given by

$$d_t = \delta \frac{z_t}{\delta + \varepsilon p_t^{1-\theta}}, \ \ e_t = \varepsilon p_t^{-\theta} \frac{z_t}{\delta + \varepsilon p_t^{1-\theta}}, \ \ c_t = \left(\delta + \varepsilon p_t^{1-\theta}\right)^{\frac{1}{\theta-1}} z_t.$$

Solving the third equation for z_t we can derive a price index, $\mathcal{P}_t$: One unit of the consumption index c_t costs

$$\mathcal{P}_t = \left(\delta + \varepsilon p_t^{1-\theta}\right)^{\frac{1}{1-\theta}}.$$

For $p_t = 1$, the price index equals unity. For $p_t \to \infty$, it increases with p_t if $\theta < 1$ but converges to a constant if $\theta > 1$. Intuitively, how strongly the relative price increase translates into a higher price index depends on the household's willingness to substitute across goods. Using the price index, we also have $d_t = \delta c_t \mathcal{P}_t^{\theta}$ and $e_t = \varepsilon c_t (\mathcal{P}_t / p_t)^{\theta}$.

Equipped with these results, we turn to the intertemporal program. The dynamic budget constraints are given by $w_0 = \mathcal{P}_0 c_0 + a_1$ and $a_1 R_1 + w_1 = \mathcal{P}_1 c_1$ where wage income and assets are expressed in terms of the numeraire d_t. The household's program therefore reads

$$\max_{c_0, c_1} u(c_0) + \beta u(c_1) \ \text{ s.t. } \ \mathcal{P}_0 c_0 + \frac{\mathcal{P}_1 c_1}{R_1} = w_0 + \frac{w_1}{R_1}$$

and the Euler equation characterizing the optimal intertemporal consumption allocation is given by

$$u'(c_0) = \beta R_1 \frac{\mathcal{P}_0}{\mathcal{P}_1} u'(c_1).$$

As usual, the marginal rate of substitution is equated with the marginal rate of transformation. But with a consumption index, the marginal rate of transformation is given by the *own rate of interest*, $R_1 \mathcal{P}_0 / \mathcal{P}_1$. The latter differs from the marginal rate of transformation for the numeraire good, R_1, whenever $\mathcal{P}_t$ (and thus, p_t) changes over time.

2.3 Bibliographic Notes

Modigliani and Brumberg (1954) and Friedman (1957) derive consumption functions from microeconomic principles. Friedman (1957) emphasizes the role of permanent as opposed to current income and Modigliani and Brumberg (1954), among others, stress life cycle considerations.

Strotz (1956), Pollak (1968), and Phelps and Pollak (1968) analyze time inconsistency and Laibson (1997) explores consequences of hyperbolic discounting. Krusell and Smith (2003) and Cao and Werning (2018) analyze determinacy of the saving function of an infinitely lived household with time inconsistent preferences. Homer (800 B.C.E., 12) describes Ulysses and the Sirens. Dixit and Stiglitz (1977) analyze a model with a CES consumption index.

Related Topics Deaton (1992) covers consumption.

Chapter 3

Dynamic Competitive Equilibrium

We now embed the consumption-saving tradeoff in two general equilibrium models of capital accumulation: We add a firm sector and impose market clearing. In the first model, we assume that all households are alike and live over the same time period. In the second model, we study the interaction between cohorts of households; households within the same cohort are alike but they differ across cohorts. In subsequent chapters, we generalize these models, for instance by introducing risk or a labor-leisure choice.

The *representative agent assumption* is convenient but restrictive. Only under specific conditions may heterogenous households with different preferences be represented by a single surrogate; see appendix B.2 for a discussion.

3.1 Representative Agent and Capital Accumulation

Our first general equilibrium model is the *representative agent* or *Ramsey model*.

3.1.1 Economy

The economy is inhabited by a continuum of identical households of mass one as well as a continuum of identical firms of mass one. Both households and firms take prices as given; households also take firm profits as given. Since households and firms are homogeneous we can represent them as a *representative household* and a *representative firm*, respectively.

3.1.2 Firms

Firms solve static profit maximization problems. In each period, they rent capital, K_t, at *rental rate* r_t and labor, L_t, at *wage* w_t from households to produce *output* (the numeraire) with a neoclassical production function, f. Profits are distributed to households.

Taking rental rates and wages as given, the representative firm maximizes

$$\max_{K_t, L_t} f(K_t, L_t) - K_t r_t - L_t w_t.$$

The first-order conditions

$$\begin{aligned} f_K(K_t, L_t) &= r_t, & (3.1) \\ f_L(K_t, L_t) &= w_t & (3.2) \end{aligned}$$

define demand functions for capital and labor. The budget constraint of the representative firm reads

$$f(K_t, L_t) = K_t r_t + L_t w_t + z_t \tag{3.3}$$

where z_t denotes profits. In equilibrium, profits equal zero, due to constant returns to scale and price taking.

3.1.3 Households

The representative household maximizes $\sum_{t=0}^{\infty} \beta^t u(c_t)$. The dynamic budget constraint and Euler equation, respectively, are given by

$$\begin{aligned} a_{t+1} &= a_t R_t + w_t - c_t + z_t, \\ u'(c_t) &= \beta R_{t+1} u'(c_{t+1}). \end{aligned}$$

Since all households are alike (and the economy is closed and there is no government sector) they do not hold claims vis-a-vis each other or third parties. Accordingly, household assets correspond to the physical capital stock in the economy: The capital stock per worker, k_t, equals a_t. Capital *depreciates* at rate δ per period. The gross *interest rate* thus equals the rental rate of capital paid by firms, r_t, plus the unit of capital net of depreciation: $R_t = 1 + r_t - \delta$.

Combining these conditions yields

$$\begin{aligned} k_{t+1} &= k_t(1 + r_t - \delta) + w_t - c_t + z_t, & (3.4) \\ u'(c_t) &= \beta(1 + r_{t+1} - \delta) u'(c_{t+1}). & (3.5) \end{aligned}$$

Households also satisfy the transversality condition $\lim_{T\to\infty} q_T k_{T+1} = 0$ or equivalently, using the Euler equation, $\lim_{T\to\infty} \beta^T u'(c_T) k_{T+1} = 0$.[1] The initial capital stock, k_0, is given.

3.1.4 Market Clearing

There are three goods in each period: Labor, capital (inherited from the last period), and output which can be used for consumption and *investment* (accumulation of new capital). Since there is one representative household whose time endowment per period equals unity, labor and capital market clearing requires firms to demand one unit of labor and k_t units of capital:

$$\begin{aligned} K_t &= k_t, & (3.6) \\ L_t &= 1. & (3.7) \end{aligned}$$

[1] With a finite horizon, the transversality condition reduces to $k_{T+1} = 0$.

By Walras' Law, market clearing in all but one market implies that the remaining market clears as well if all agents satisfy their budget constraints. In the economy considered here, this can be seen by combining (3.3), (3.4), (3.6), and (3.7) to find

$$k_{t+1} = k_t(1 + r_t - \delta) + w_t - c_t + f(k_t, 1) - k_t r_t - 1w_t$$

which simplifies to the *resource constraint*

$$k_{t+1} = k_t(1 - \delta) + f(k_t, 1) - c_t.$$

The resource constraint states that the market for the output good clears. Equivalently, *gross investment* plus consumption equals output, or saving equals *net investment*. The condition represents the GDP identity in a closed economy without government sector.

3.1.5 General Equilibrium

In general equilibrium, the transversality condition as well as conditions (3.1)–(3.7) and thus, the resource constraint hold at all dates. The equilibrium conditions can be reduced to (i) the transversality condition; (ii) two core equations in capital and consumption; and (iii) five remaining conditions that determine r_t, w_t, z_t, K_t, and L_t. The two core equations are given by the resource constraint and the Euler equation with the rental rate of capital expressed in terms of the marginal product of capital:

$$\begin{aligned} k_{t+1} &= k_t(1 - \delta) + f(k_t, 1) - c_t, && (3.8) \\ u'(c_t) &= \beta(1 + f_K(k_{t+1}, 1) - \delta)u'(c_{t+1}). && (3.9) \end{aligned}$$

Note that, conditional on k_t and c_t, these two equations pin down k_{t+1} and c_{t+1}.

For a given initial capital stock, k_0, conditions (3.8) and (3.9) completely pin down the equilibrium sequences for capital and consumption once a starting value for consumption, c_0, is specified. This starting value cannot freely be chosen, however, because the sequences also must satisfy the transversality condition,

$$\lim_{T \to \infty} \beta^T u'(c_T) k_{T+1} = 0.$$

As we will see below, there is a unique c_0 such that the paths implied by (k_0, c_0) as well as (3.8) and (3.9) satisfy the transversality condition.

3.1.6 Social Planner Allocation and Pareto Optimality

We have characterized the equilibrium conditions in the *decentralized economy* with firms and households. Alternatively, we can characterize the *social planner allocation* in a *Robinson Crusoe economy*. This economy is inhabited by a single consumer-producer who operates the production function f and saves in the form of capital. Robinson Crusoe solves

$$\max_{\{c_t, k_{t+1}\}_{t \geq 0}} \sum_{t=0}^{\infty} \beta^t u(c_t) \text{ s.t. } k_{t+1} = k_t(1 - \delta) + f(k_t, 1) - c_t, \ k_0 \text{ given}, \ k_{t+1} \geq 0.$$

The non-negativity constraint on capital is not binding if f satisfies the Inada conditions. Solving this program yields exactly the same conditions as those characterizing the decentralized equilibrium, namely conditions (3.8) and (3.9) and the transversality condition (see appendix B.3).

Since the social planner allocation is the feasible allocation preferred by the representative household it necessarily is Pareto optimal. By implication, the decentralized equilibrium allocation is Pareto optimal as well. This is not surprising since the economy satisfies the conditions of the first welfare theorem.

In fact, given that these conditions are satisfied, we could have directly characterized the equilibrium allocation (and also prices) by solving the *social planner problem*. This holds true much more generally. As long as the conditions of the welfare theorems are satisfied, the set of Pareto optimal allocations corresponds to the set of competitive equilibria, possibly subject to transfers between different groups of households. Rather than solving the conditions characterizing the decentralized equilibrium we may therefore find the equilibrium allocation directly by solving a corresponding social planner problem (see subsection 8.3.3 for an application in a model with heterogeneous households).

3.1.7 Analysis

Phase Diagram

Since (3.8) and (3.9) constitute non-linear first-order difference equations the model cannot generally be solved in closed form. However, we may qualitatively characterize equilibrium by means of a *phase diagram* which illustrates the system dynamics. The phase diagram is constructed based on the relations

$$c_t = f(k_t, 1) - \delta k_t, \tag{3.10}$$

$$1 = \beta(1 + f_K(k_{t+1}, 1) - \delta). \tag{3.11}$$

Condition (3.10) follows from (3.8) when the capital stock is constant over time, $k_t = k_{t+1}$. In this case, consumption plus *replacement investment* equals output. Condition (3.11) follows from (3.9) when $c_t = c_{t+1}$. For consumption to be constant over time, βR_{t+1} must equal unity.

In figure 3.1, relations (3.10) and (3.11) are represented by the black concave schedule and the black vertical line, respectively. Their intersection indicates the *steady state* of the system, (k, c), where all equilibrium conditions are satisfied and all variables do not change over time.

From (3.10), consumption is maximized subject to a time invariant capital stock when the latter equals the *golden-rule* capital stock, k^{gr}, which satisfies

$$f_K(k^{\text{gr}}, 1) = \delta.$$

The steady-state or *modified-golden-rule* capital stock is lower than the golden-rule capital stock, $k < k^{\text{gr}}$, because from (3.11)

$$f_K(k, 1) = \delta + \beta^{-1} - 1 > \delta.$$

Capital stock dynamics outside of steady state are determined by the resource constraint. Suppose that $c_t > f(k_t, 1) - \delta k_t$ that is, c_t lies above the concave schedule. Gross investment then falls short of the replacement investment necessary to maintain the capital stock, and as a consequence $k_{t+1} < k_t$. Conversely, a choice of c_t below the concave schedule implies $k_{t+1} > k_t$.

Consumption dynamics outside of steady state are determined by the Euler equation. If the capital stock is smaller than k then the marginal product of capital and thus, the gross rate of return on capital are higher than in steady state and consumption rises, $c_{t+1} > c_t$. Conversely, a capital stock larger than k implies $c_{t+1} < c_t$.

The system dynamics therefore differ across the four regions separated by (3.10) and (3.11): If $k_t < k$ and $c_t < f(k_t, 1) - \delta k_t$ then both the capital stock and consumption rise over time. If $k_t > k$ and $c_t < f(k_t, 1) - \delta k_t$ then the capital stock rises and consumption falls. If $k_t < k$ and $c_t > f(k_t, 1) - \delta k_t$ then the capital stock falls and consumption rises. Finally, if $k_t > k$ and $c_t > f(k_t, 1) - \delta k_t$ then both the capital stock and consumption fall.

The paths indicated by dots and arrows in figure 3.1 illustrate the system dynamics. Consider a low initial capital stock, $k_0 = 0.5k$ say. The figure illustrates three candidate adjustment paths that start at different initial consumption levels, c_0. All these candidate paths satisfy (3.8) and (3.9) but only one—the path towards the steady state indicated by dots—satisfies (3.8) and (3.9) and meets the transversality condition. Too low an initial consumption level implies non-convergent dynamics to the bottom right (indicated by arrows) where the interest rate is negative and thus, the transversality condition violated. Too high an initial consumption level implies non-convergent dynamics to the top left (also indicated by arrows) where the Euler equation prescribes consumption growth but household assets tend to zero.

Similarly, for a high initial capital stock, $k_0 = k^{gr}$ say, too low or too high an initial consumption value implies non-convergent dynamics, indicated by the paths marked with arrows that start above k^{gr}, while an appropriate intermediate starting value implies convergent dynamics, indicated by the dotted path towards the steady state.

As a function of the initial capital stock, k_0, the consumption level guaranteeing convergent equilibrium dynamics, $c_0(k_0)$, traces out the *saddle path*. The saddle path gives the equilibrium initial consumption level for an initial capital stock, and it indicates the path along which convergent equilibrium dynamics occur. The dotted path in figure 3.1 illustrates the segment of the saddle path between $0.5k$ and k^{gr}.

Based on the phase diagram we may not only analyze equilibrium dynamics in environments with constant technology (and preferences) but also the response to changing fundamentals. Suppose, for example, that the production function is known to change from f to g say at some future date $t = T$. From that future date onwards, system dynamics are governed by the Euler equation and resource constraint associated with g rather than f. Before date $t = T$, in contrast, the Euler equation and resource constraint associated with f determine the dynamic behavior.

Capital and consumption do not abruptly move at date $t = T$ from the old to the new saddle path—this would violate the Euler equation or the resource constraint. Instead, for $t < T$, (k_t, c_t) moves *off* the old saddle path towards the new saddle path;

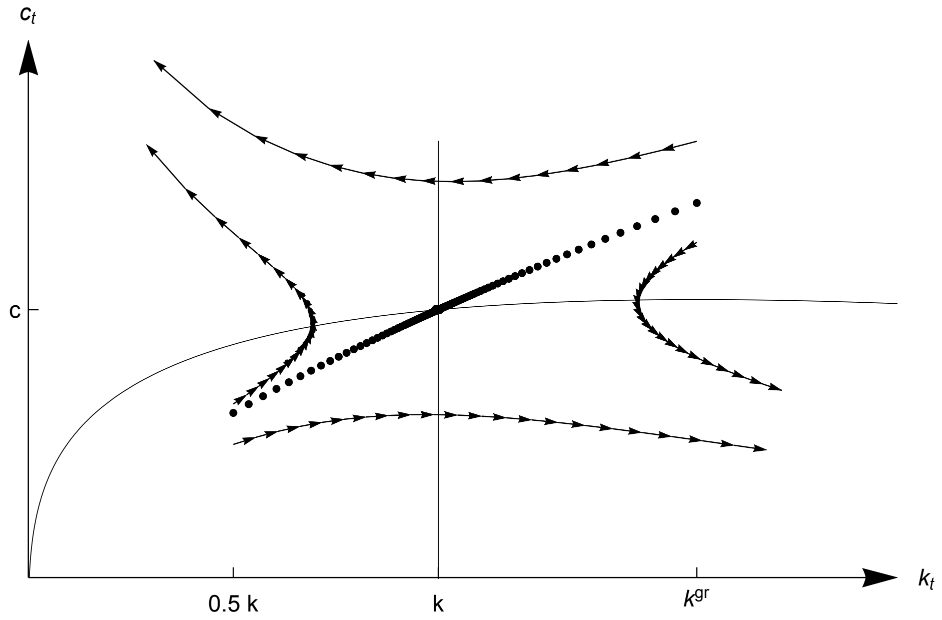

Figure 3.1: Dynamics in the representative agent model: The saddle path is indicated by dots.

at date $t = T$, (k_t, c_t) meets the new saddle path; and for $t > T$, (k_t, c_t) moves along the new saddle path towards the long-run steady state which is determined by the new Euler equation and resource constraint.

Solution Methods

Beyond the phase diagram, several strategies may be used to solve the model. One, described below, is based on a *linear approximation* of the difference equation system (3.8) and (3.9); it involves eigenvalue and eigenvector operations. Another strategy is based on simply trying different starting values c_0 conditional on k_0 and checking whether the induced system dynamics are convergent. Finally, one may solve the social planner's program numerically, using dynamic programming methods.

The approximation strategy rests on linearizing (3.8) and (3.9) around the steady state. For example, totally differentiating (3.8) and evaluating at the steady state yields

$$
\begin{aligned}
& dk_{t+1} + dc_t = (f_K(k, 1) + 1 - \delta) dk_t, \\
\Rightarrow \quad & \hat{k}_{t+1} + \hat{c}_t \frac{c}{k} = \beta^{-1} \hat{k}_t,
\end{aligned}
$$

where a circumflex denotes infinitesimal relative deviations from the corresponding steady state value, e.g., $\hat{c}_t \equiv (c_t - c)/c$. Similarly, taking logarithms in (3.9), totally differentiating and evaluating at the steady state yields (letting $\sigma \equiv -u''(c)c/u'(c)$)

$$\begin{aligned}
& \ln(u'(c_t)) = \ln(\beta) + \ln(1 + f_K(k_{t+1}, 1) - \delta) + \ln(u'(c_{t+1})), \\
\Rightarrow \quad & -\sigma \frac{dc_t}{c} = \frac{f_{KK}(k,1)dk_{t+1}}{1 + f_K(k,1) - \delta} - \sigma \frac{dc_{t+1}}{c}, \\
\Rightarrow \quad & \hat{c}_t = -\frac{\beta}{\sigma} f_{KK}(k,1)\, k\, \hat{k}_{t+1} + \hat{c}_{t+1}.
\end{aligned}$$

Approximating the original system to the first order means that we apply the linearized system to deviations from steady state even if they are larger than infinitesimal.

Next, we represent the linearized equations in vector and matrix form as

$$M_1 \begin{bmatrix} \hat{c}_{t+1} \\ \hat{k}_{t+1} \end{bmatrix} = M_0 \begin{bmatrix} \hat{c}_t \\ \hat{k}_t \end{bmatrix}, \; M_1 \equiv \begin{bmatrix} 1 & -\frac{\beta}{\sigma} f_{KK}(k,1)k \\ 0 & 1 \end{bmatrix}, \; M_0 \equiv \begin{bmatrix} 1 & 0 \\ -\frac{c}{k} & \beta^{-1} \end{bmatrix}.$$

Multiplying by the inverse of M_1 yields

$$\begin{bmatrix} \hat{c}_{t+1} \\ \hat{k}_{t+1} \end{bmatrix} = M \begin{bmatrix} \hat{c}_t \\ \hat{k}_t \end{bmatrix}, \; M \equiv M_1^{-1} M_0 = \begin{bmatrix} 1 - \frac{c}{\sigma}\beta f_{KK} & \frac{f_{KK}k}{\sigma} \\ -\frac{c}{k} & \beta^{-1} \end{bmatrix}.$$

Finally, we express the linearized system in terms of the eigenvalues ρ_1 and ρ_2 as well as the corresponding eigenvectors v_1 and v_2 of the matrix M. An eigenvalue of M satisfies $\det(M - \rho I) = 0$ that is, it solves the characteristic equation $\mathcal{C}(\rho) = 0$ with $\mathcal{C}(\rho) \equiv \rho^2 - \rho(1 + \beta^{-1} - c\beta f_{KK}(k,1)/\sigma) + \beta^{-1}$. The latter is a continuous quadratic function satisfying $\mathcal{C}(0) > 0$, $\mathcal{C}(1) < 0$, and $\lim_{\rho \to \infty} \mathcal{C}(\rho) = \infty$. It follows that $0 < \rho_1 < 1 < \rho_2$. In fact, $\rho_2 = 1/(\beta\rho_1)$. Using standard results (see appendix A.3), we have

$$\begin{bmatrix} \hat{c}_t \\ \hat{k}_t \end{bmatrix} = \varphi_1 \rho_1^t v_1 + \varphi_2 \rho_2^t v_2,$$

where φ_1, φ_2 are arbitrary constants that remain to be determined. The requirement that system dynamics be stable implies that φ_2 must equal zero since ρ_2^t grows without bound as $t \to \infty$. The second constant, φ_1, is pinned down by the initial condition for the capital stock, $\hat{k}_0 = \varphi_1 v_{1[2]}$, where $v_{1[2]}$ denotes the second element of the eigenvector v_1. In conclusion,

$$\begin{bmatrix} \hat{c}_t \\ \hat{k}_t \end{bmatrix} = \rho_1^t v_1 \frac{\hat{k}_0}{v_{1[2]}}.$$

The saddle path of the linearized system is given by the function

$$\hat{c}_0(\hat{k}_0) = \frac{\hat{k}_0}{v_{1[2]}} v_{1[1]}.$$

Its slope in (k, c)-space satisfies

$$\frac{dc_0}{dk_0} = \frac{c}{k} \frac{v_{1[1]}}{v_{1[2]}}.$$

For $\sigma \to \infty$ or $f_{KK}k/f_K \to 0$, the slope approaches $(1-\beta)/\beta$. Lower values of σ (a higher intertemporal elasticity of substitution) or more negative elasticities of the marginal product with respect to k increase dc_0/dk_0.

The *speed of convergence* to the steady state is determined by the stable eigenvalue, ρ_1. The higher this eigenvalue, the slower the convergence.

3.1.8 Population Growth

Suppose the number of household members grows at gross rate ν per period and the household's objective thus equals $\sum_{t=0}^{\infty} \beta^t \nu^t u(c_t)$ where c_t denotes per-capita consumption as before. Normalizing the population size at date $t = 0$ to unity, the resource constraint now is given by

$$\nu^{t+1} k_{t+1} = \nu^t k_t (1-\delta) + f(\nu^t k_t, \nu^t) - \nu^t c_t,$$

where k_t continues to denote the capital stock per capita. Equivalently,

$$\nu k_{t+1} = k_t(1-\delta) + f(k_t, 1) - c_t.$$

Intuitively, positive population growth implies that the capital-labor ratio at date $t+1$ is smaller (by the factor ν) than the per-capita resources not consumed at date t. Except for this difference, the conditions characterizing the centralized or decentralized equilibrium are not affected by population growth.

3.2 Overlapping Generations and Capital Accumulation

Our second general equilibrium model is the *overlapping generations model*.

3.2.1 Economy

There is an infinite number of two-period lived cohorts. Subsequent generations overlap, see figure 3.2: In each period, a continuum of young and old households of mass one each inhabit the economy. Young households are born without assets; they work, consume, and save for retirement. Old households retire, consume the return on their saving (i.e., households leave no bequests), and die. The assets held by the retirees correspond to the capital stock in the economy. At date $t = 0$ the old cohort is endowed with the initial capital stock, k_0.

3.2.2 Firms

The firm sector is identical to the one in the representative agent model and conditions (3.1)–(3.3) apply. Profits are distributed to old households.

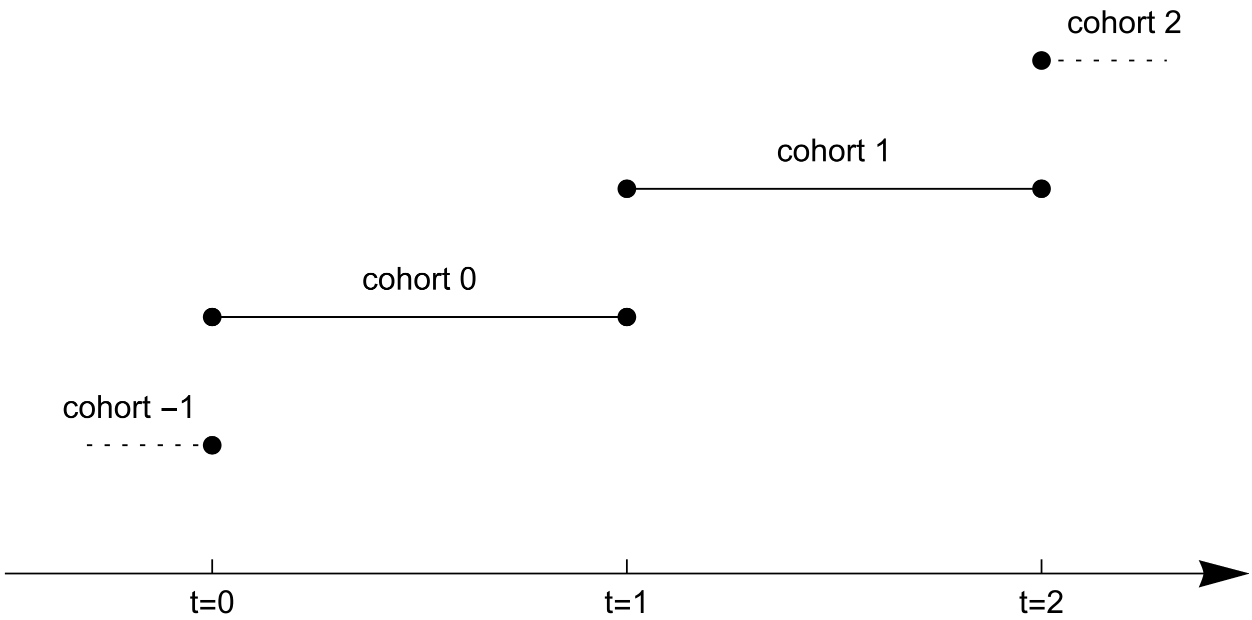

Figure 3.2: Overlapping generations.

3.2.3 Households

The dynamic budget constraints of a worker and a retiree at date t as well as the Euler equation of a young household are given by

$$k_{t+1} = w_t - c_{1,t}, \quad (3.12)$$
$$c_{2,t} = k_t(1 + r_t - \delta) + z_t, \quad (3.13)$$
$$u'(c_{1,t}) = \beta(1 + r_{t+1} - \delta)u'(c_{2,t+1}), \quad (3.14)$$

respectively. Here, $c_{1,t}$ and $c_{2,t}$ denote consumption at date t of a young and old household, respectively.

3.2.4 Market Clearing

Labor and capital market clearing requires that firms demand one unit of labor and k_t units of capital, implying the equilibrium conditions (3.6) and (3.7). By Walras' Law, market clearing in all but one market implies that the remaining market clears as well if all agents satisfy their budget constraints. Combining (3.3), (3.6), (3.7), (3.12), and (3.13) and letting $c_t \equiv c_{1,t} + c_{2,t}$ yields

$$\begin{aligned} k_{t+1} &= w_t - c_{1,t} + k_t(1 + r_t - \delta) - c_{2,t} + z_t \\ &= k_t(1 + r_t - \delta) + w_t - c_t + f(k_t, 1) - k_t r_t - 1 w_t. \end{aligned}$$

This simplifies to the same resource constraint as in the representative agent model,

$$k_{t+1} = k_t(1 - \delta) + f(k_t, 1) - c_t.$$

3.2.5 General Equilibrium

In general equilibrium, conditions (3.1)–(3.3), (3.6)–(3.7), and (3.12)–(3.14) (and thus, the resource constraint) hold simultaneously. These equilibrium conditions can be reduced to three core equations,

$$\begin{aligned} k_{t+1} &= k_t(1-\delta) + f(k_t, 1) - c_{1,t} - c_{2,t}, \\ c_{2,t} &= k_t(1 + f_K(k_t, 1) - \delta), \\ u'(c_{1,t}) &= \beta(1 + f_K(k_{t+1}, 1) - \delta) u'(c_{2,t+1}), \end{aligned}$$

as well as five remaining conditions that determine r_t, w_t, z_t, K_t, and L_t. Compared with the representative agent model, an additional budget constraint is present; it determines how consumption is split between workers and retirees. The Euler equation characterizes the slope of the consumption profile over the household's life cycle.

Conditional on k_t, the second core equation pins down $c_{2,t}$. Moreover, since $c_{2,t+1} = k_{t+1}(1 + f_K(k_{t+1}, 1) - \delta)$, the first and third condition pin down $c_{1,t}$ and k_{t+1}. For an initial capital stock, k_0, the core equations therefore completely determine the equilibrium paths of capital and consumption over the infinite horizon.

An alternative representation of equilibrium uses the *saving function*. Let $a_{t+1} = a(w_t, R_{t+1})$ denote equilibrium saving of a worker. The saving function a combines the Euler equation and the intertemporal budget constraint which extends over two periods; it depends on wealth (given by the wage) and the interest rate. Combined with the equilibrium relations between factor prices and the capital-labor ratio, the saving function defines a *law of motion* for capital,

$$k_{t+1} = a(w_t, R_{t+1}) \text{ where } w_t = f_L(k_t, 1),\ R_{t+1} = 1 - \delta + f_K(k_{t+1}, 1). \tag{3.15}$$

Under certain functional form assumptions this law of motion can be solved in closed form.

Depending on preferences and technology the function $k_{t+1}(k_t)$ defined by (3.15) may intersect the 45 degree line never, once, or multiple times; accordingly, no steady state with a strictly positive capital stock, a unique such steady state, or multiple steady states may exist. A steady state is stable and non-oscillating if in a neighborhood around it, k_{t+1} increases in k_t, but by less than one-to-one. Writing (3.15) as $k_{t+1} = \tilde{a}(k_t, k_{t+1})$ and totally differentiating implies

$$\frac{dk_{t+1}}{dk_t} = \frac{\partial \tilde{a}(k_t, k_{t+1}) / \partial k_t}{1 - \partial \tilde{a}(k_t, k_{t+1}) / \partial k_{t+1}}.$$

A steady state thus is stable and non-oscillating if the value of the expression on the right-hand side, evaluated at steady state, lies between zero and one.

3.2.6 Analysis

In contrast to the representative agent model, households in the overlapping generations model are heterogeneous. As a consequence, average saving in the economy differs from the saving of young or old households, and the slope of the consumption profile of a young household need not match the slope of the aggregate consumption profile.

This has important implications for the steady state. While the first steady-state condition of the representative agent model, condition (3.10), also applies in the overlapping generations model, the second one, condition (3.11), does not. In the representative agent model, this second condition follows from the requirement that aggregate and thus, individual consumption is constant over time. In the overlapping generations model, in contrast, constancy of aggregate consumption (or of young-age consumption or old-age consumption) does not imply that the consumption profile of an individual household is flat over the life cycle. The steady-state capital stock, the fixed point of (3.15), therefore need not satisfy the condition $\beta(1-\delta+f_K(k,1))=1$. In fact, depending on preferences and the production function, it can be smaller or—unlike in the representative agent model—larger than the golden-rule capital stock.

3.2.7 Pareto Optimality

Since the steady-state capital stock in the overlapping generations model may exceed the golden-rule capital stock, the steady-state interest rate need not satisfy $R > 1$. This contrasts sharply with the situation in the representative agent model where the steady-state net interest rate always is positive, and it has important efficiency implications. In fact, a steady state with $R < 1$ is *Pareto inefficient*, for two reasons.

First, because the economy *over accumulates capital* or is *dynamically inefficient*. When $R < 1$ the marginal unit of capital contributes negatively to steady-state total consumption: The marginal contribution to output, $f_K(k,1)$ which equals r in equilibrium, falls short of the marginal replacement investment to compensate for depreciation, δ. When $R < 1$, a reduction of the capital stock therefore does not only free resources for contemporaneous consumption; when the capital stock is permanently reduced, it also frees resources in all future periods because the fall in output is more than compensated by lower replacement investment. Accordingly, a steady-state allocation with $R < 1$ (that is, $r < \delta$) cannot be Pareto optimal.

To take a stark example, suppose that capital does not contribute to production at all ($f_K(K_t, L_t) = 0$, capital accumulation amounts to storage) and depreciates at a positive rate (a fraction of the stored goods spoils). The condition for dynamic inefficiency then is met and from a social perspective, which only takes feasibility restrictions into account, capital accumulation is wasteful. Nevertheless, households do accumulate capital to finance old-age consumption because they must satisfy their budget constraints.

Second, a steady state with $R < 1$ also is inefficient because it entails a *suboptimal allocation of consumption* over the life cycle. To see this, consider an endowment economy with a fixed total endowment in each period and suppose that the steady-state interest rate satisfies $R < 1$. A transfer scheme that takes Δ units of the good from each young household and gives them to each old household then makes everybody better off: The old in the period when the scheme is introduced gain because they receive Δ

without having to contribute; and the young in this and all subsequent periods gain because the effective gross interest rate of unity which they receive on their contribution exceeds the market interest rate. (See also the discussions in subsection 5.3.2 and section 9.2.)

When households save both sources of inefficiency are present. Going back to the example of storage with depreciation assume that storage k pays a rate of return $1-\delta < 1$. With a young-age endowment w and transfers Δ, young-age consumption is given by $w-k-\Delta$ and old-age consumption equals $k(1-\delta)+\Delta$. A marginal increase of Δ improves welfare of current and future generations as long as $-u'(w-k-\Delta)+\beta u'(k(1-\delta)+\Delta) \geq 0$ (using the envelope condition). But from the Euler equation, $-u'(w-k-\Delta)+\beta(1-\delta)u'(k(1-\delta)+\Delta)=0$ as long as households store. We conclude that an expansion of the transfer scheme improves welfare of all generations as long as the Euler equation holds with equality that is, as long as households store.

The possibility of Pareto improving transfers in an inefficient economy hinges on the assumption of an infinite horizon. If there existed a last period then transferring resources from the young to the old would hurt the young in the last period and the transfer scheme would not lead to a Pareto improvement. Related, in an inefficient economy the market value of endowments is infinite (since $R < 1$ and the endowment sequence has infinite length), reflecting the *double infinity* of households and commodities. The proof of the first welfare theorem which relies on a finite market value of endowments therefore does not go through. Even if all cohorts could trade with each other (i.e., all cohorts were infinitely lived but for all $t \geq 0$, cohort t only valued consumption at dates t and $t+1$) a steady state with $R < 1$ still could arise—and still would be inefficient.

3.2.8 Population Growth

Suppose the number of young households grows at gross rate ν per period and normalize the cohort size at date $t=0$ to unity. Maintaining the definition of k_t as capital stock per worker as well as $c_{1,t}$ and $c_{2,t}$ as per-capita consumption, the budget constraints in equilibrium now read

$$\begin{aligned} c_{1,t} &= w_t - k_{t+1}\nu, \\ c_{2,t} &= k_t(1+r_t-\delta)\nu \end{aligned}$$

and the resource constraint is given by

$$\nu^{t+1}k_{t+1} = \nu^t k_t(1-\delta) + f(\nu^t k_t, \nu^t) - \nu^t c_{1,t} - \nu^{t-1}c_{2,t}$$

or

$$\nu k_{t+1} = k_t(1-\delta) + f(k_t, 1) - c_{1,t} - c_{2,t}/\nu.$$

The condition for inefficiency generalizes to

$$f_K(k,1) < \delta + \nu - 1 \text{ or } R < \nu,$$

relating the net marginal product of capital or the interest rate to the net growth rate of the economy.

3.3 Bibliographic Notes

The representative agent, Ramsey, or neoclassical growth model is due to Ramsey (1928), Cass (1965), and Koopmans (1965). According to Fisher's (1930) *separation theorem* a firm's objective is to maximize the present discounted value of profits, independently of the owners' preferences and of financing decisions. Negishi (1960) characterizes competitive equilibrium by solving a social planner problem.

The overlapping generations model builds on Allais (1947) and is due to Samuelson (1958); Diamond (1965) introduces capital in the model. Modigliani and Brumberg (1954) discuss life cycle saving as well as the aggregation of heterogeneous consumption profiles. Shell (1971) and Balasko and Shell (1980) analyze inefficiency in overlapping generations endowment economies. Malinvaud (1953), Phelps (1965), and Cass (1972) analyze capital over accumulation.

Related Topics Gale (1973) analyzes overlapping generations endowment economies and studies determinacy of the equilibrium allocation; see also Kehoe and Levine (1985) on indeterminacy and Grandmont (1985) on endogenous cycles in overlapping generations economies.

Chattopadhyay and Gottardi (1999) analyze inefficiency in overlapping generations economies with endowment risk. Zilcha (1990) analyzes capital over accumulation in environments with stochastic production; see also Barbie, Hagedorn and Kaul (2007).

Yaari (1965) and Blanchard (1985) analyze models of *perpetual youth* where households face a constant probability of death while new cohorts enter the economy.

See chapter 6 for models of sunspot-driven business cycles and chapter 9 for money in the overlapping generations model.

Chapter 4

Risk

With *risk*, income and consumption may be random that is, vary across histories at a given date. This introduces new elements in the household's consumption-saving tradeoff. We study this tradeoff in two environments, with incomplete markets and complete markets respectively, depending on the number of assets with linearly independent returns. Thereafter, we analyze risk sharing and study how uninsurable income risk affects capital accumulation. Throughout, we assume that households evaluate random consumption according to the expected utility criterion; in appendix B.4 we discuss a model that relaxes this assumption.

4.1 Consumption, Saving, and Insurance

4.1.1 Incomplete Markets

Suppose that there are two periods. In the second period, one of two histories is realized, $\epsilon^1 = h$ or $\epsilon^1 = l$, with probability $\eta(h)$ and $\eta(l)$ respectively. The wage in the second period, $w_1(\epsilon^1)$, varies by history; it equals $w_1(h)$ or $w_1(l)$. The household saves in one asset, a_1, and maximizes expected present discounted utility from consumption, $u(c_0) + \beta\mathbb{E}_0[u(c_1(\epsilon^1))]$. The program reads

$$\begin{aligned}\max_{a_1, c_0, c_1(h), c_1(l)} \quad & u(c_0) + \beta(\eta(h)u(c_1(h)) + \eta(l)u(c_1(l))) \\ \text{s.t.} \quad & a_1 = w_0 - c_0, \;\; c_1(\epsilon^1) = a_1 R_1 + w_1(\epsilon^1).\end{aligned}$$

Note that the return on saving is not history-contingent, in contrast to the wage. This assumption is not critical. What is crucial—and renders *markets incomplete*—is that fewer assets than states of nature are available. Note also that consumption in the second period is history-contingent.

Substituting the second-period dynamic budget constraints into the first-period dynamic budget constraint, we find one intertemporal budget constraint for each history,

$$c_0 + \frac{c_1(h)}{R_1} = w_0 + \frac{w_1(h)}{R_1} \text{ and } c_0 + \frac{c_1(l)}{R_1} = w_0 + \frac{w_1(l)}{R_1}.$$

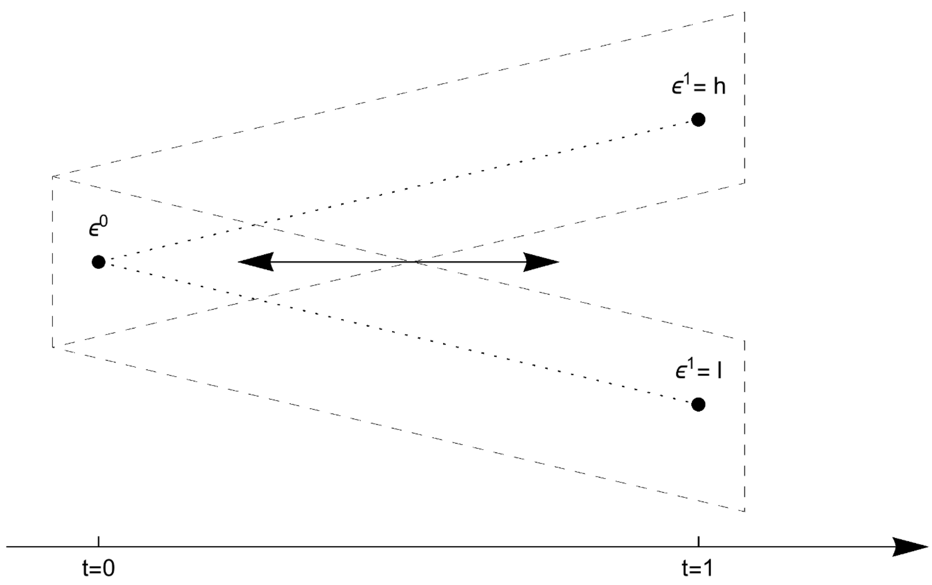

Figure 4.1: Incomplete markets: Two intertemporal budget constraints and one adjustment margin.

If the second-period wage assumes the high value, then lifetime consumption expenditures discounted at the interest rate R_1 must equal $w_0 + w_1(h)/R_1$. If it assumes the low value then they must equal $w_0 + w_1(l)/R_1$.

The household faces incomplete markets because it cannot exchange consumption in history h against consumption in history l. Figure 4.1 illustrates this. The dashed parallelograms indicate the range of the two intertemporal budget constraints: One connects the initial period and the high state in the second period, the other the initial period and the low state. The arrows indicate the household's single margin of adjustment, corresponding to the choice of a_1: Purchasing power can be shifted over time—saving reduces c_0 and increases *both* $c_1(h)$ and $c_1(l)$—but it cannot be shifted across nodes at date $t = 1$.

To characterize the solution to the household's problem we can substitute the dynamic budget constraints into the objective function and differentiate with respect to a_1. This yields the *stochastic Euler equation*

$$u'(c_0) = \beta R_1 \mathbb{E}_0 \left[u'(c_1(\epsilon^1)) \right].$$

Intuitively, the cost of saving represented on the left-hand side is balanced with the average benefit across histories represented on the right-hand side.

Precautionary Saving

Assume that $\beta R_1 = 1$ such that the Euler equation reduces to $u'(c_0) = \mathbb{E}_0 \left[u'(c_1(\epsilon^1)) \right]$. Without risk, the equilibrium consumption profile would be flat in this case. With

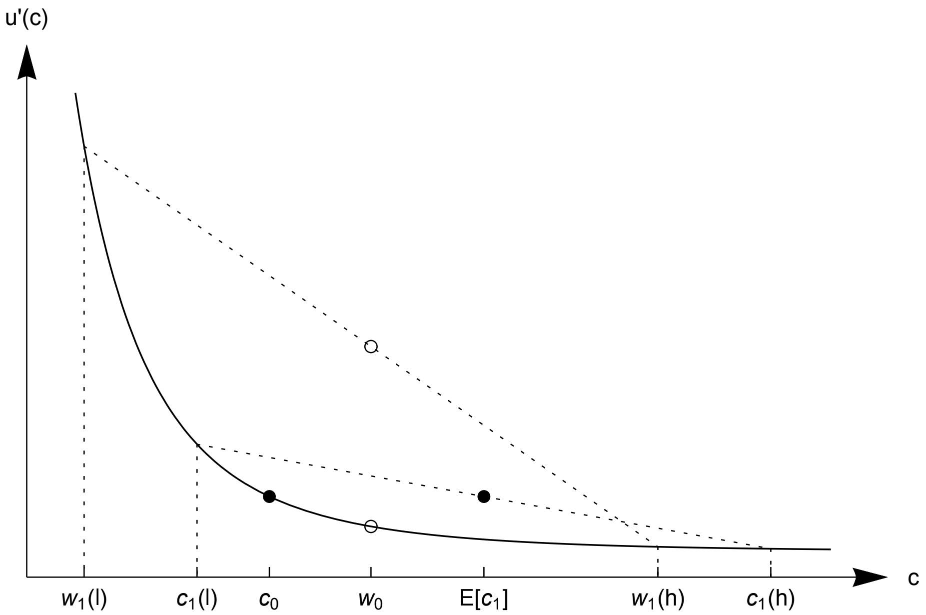

Figure 4.2: Precautionary saving: Convex marginal utility and income risk.

risk, in contrast, it cannot be flat for all ϵ^1 because $w_1(\epsilon^1)$ is stochastic; in fact, the consumption profile generally is not even flat on average. To see this, assume that preferences are not only strictly concave, $u' > 0, u'' < 0$, as usual, but marginal utility also is convex, $u''' > 0$. Most commonly used period utility functions satisfy this condition. By Jensen's inequality we then have $\mathbb{E}_0 \left[u'(c_1(\epsilon^1))\right] > u'(\mathbb{E}_0 \left[c_1(\epsilon^1)\right])$ and the Euler equation therefore stipulates $u'(c_0) > u'(\mathbb{E}_0 \left[c_1(\epsilon^1)\right])$ or $c_0 < \mathbb{E}_0 \left[c_1(\epsilon^1)\right]$. We conclude that convex marginal utility implies strictly positive average consumption growth, in spite of $\beta R_1 = 1$; saving is higher than in the absence of risk, reflecting a *precautionary saving* motive or *prudence*. In contrast, linear marginal utility would imply $c_0 = \mathbb{E}_0 \left[c_1(\epsilon^1)\right]$ and concave marginal utility $c_0 > \mathbb{E}_0 \left[c_1(\epsilon^1)\right]$.

Figure 4.2 illustrates the precautionary saving motive. The solid line indicates the marginal utility function. Suppose first that the wage is deterministic and equal to w_0 in both periods. Since $\beta R_1 = 1$ optimal consumption then equals w_0 in both periods and saving equals zero. Consider next the case with a risky wage in the second period, $w_1(h)$ or $w_1(l)$. If the household does not save then first-period consumption equals w_0 and history-contingent second-period consumption equals $w_1(l)$ or $w_1(h)$. Due to the convexity of the marginal utility function, expected marginal utility of second-period consumption, indicated by the upper circle, exceeds marginal utility of first-period consumption, indicated by the lower circle, and the Euler equation is violated. Intuitively, the downside risk affects average marginal utility more strongly than the upside. To satisfy the Euler equation, saving must rise, first-period consumption must fall to c_0, and history-contingent second-period consumption must rise to $c_1(l)$ and $c_1(h)$. In equilibrium, marginal utility in the first period and expected marginal utility in the second period, indicated by the black dots, coincide.

Certainty Equivalence

Strict convexity of the marginal utility function in combination with risky consumption renders it difficult to solve the model. A linear marginal utility specification (quadratic utility) circumvents this problem—at the cost of doing away with the precautionary saving motive—because it implies $\mathbb{E}_0[u'(c_1(\epsilon^1))] = u'(\mathbb{E}_0[c_1(\epsilon^1)])$ and thus, the Euler equation $u'(c_0) = \beta R_1 u'(\mathbb{E}_0[c_1(\epsilon^1)])$. Note that the latter equation corresponds to the condition in the deterministic case except that risk-free second-period consumption is replaced by the conditional expectation of risky second-period consumption. The fact that the equilibrium condition is linear and only contains the conditional mean (rather than also higher-order moments) of the random variable is referred to as *certainty equivalence*.

To appreciate the gain in tractability due to certainty equivalence, consider a three-period model with quadratic utility and $\beta R_t = 1$ at all times. The intertemporal budget constraint along each history reads

$$c_0 + \beta c_1(\epsilon^1) + \beta^2 c_2(\epsilon^2) = w_0 + \beta w_1(\epsilon^1) + \beta^2 w_2(\epsilon^2)$$

and the Euler equations imply that consumption follows a *Martingale*—the conditional expectation of the change of consumption equals zero: $c_0 = \mathbb{E}_0[c_1(\epsilon^1)]$ and $c_1(\epsilon^1) = \mathbb{E}_1[c_2(\epsilon^2)]$. Using the law of iterated expectations, we can combine these results to find

$$c_0(1 + \beta + \beta^2) = w_0 + \beta \mathbb{E}_0[w_1(\epsilon^1)] + \beta^2 \mathbb{E}_0[w_2(\epsilon^2)].$$

At date $t = 1$, history ϵ^1, the intertemporal budget constraint conditional on saving in the initial period ($a_1 = w_0 - c_0$) reads

$$c_1(\epsilon^1) + \beta c_2(\epsilon^2) = (w_0 - c_0)\beta^{-1} + w_1(\epsilon^1) + \beta w_2(\epsilon^2),$$

and the Euler equation is given by $c_1(\epsilon^1) = \mathbb{E}_1[c_2(\epsilon^2)]$. Taking expectations and combining the two conditions yields

$$c_1(\epsilon^1)(1 + \beta) = (w_0 - c_0)\beta^{-1} + w_1(\epsilon^1) + \beta \mathbb{E}_1[w_2(\epsilon^2)].$$

Comparing the results for c_0 conditional on information at date $t = 0$ and for $c_1(\epsilon^1)$ conditional on information at date $t = 1$, history ϵ^1, we note that

$$(c_1(\epsilon^1) - c_0)(1 + \beta) = (\mathbb{E}_1 - \mathbb{E}_0)[w_1(\epsilon^1) + \beta w_2(\epsilon^2)].$$

We conclude that the sign and magnitude of the innovation $c_1(\epsilon^1) - \mathbb{E}_0[c_1(\epsilon^1)]$ reflects how the expected present discounted value of income in and after date $t = 1$ changes as the information set changes from date $t = 0$ to date $t = 1$, history ϵ^1.

Risk of Binding Borrowing Constraint

A binding borrowing constraint reduces consumption. It also affects consumption earlier in time, before the constraint binds. In a stochastic environment, this effect is present whenever a borrowing constraint may bind with strictly positive probability. We illustrate this in a three-period model with stochastic income $w_1(\epsilon^1)$ in the second period and non-stochastic income w_0 and w_2 otherwise. For simplicity, we let $\beta = 1$ and assume that the gross interest rate also equals unity. Only borrowing at date $t = 1$ is prohibited, $a_2(\epsilon^1) \geq 0$.[1]

We start by deriving the value function at date $t = 1$ when uncertainty is resolved. In histories where $a_1 + w_1(\epsilon^1) \geq w_2$ the preferred level of a_2 is positive and the borrowing constraint does not bind. Consumption in the second and third period is equal in this case and given by $(a_1 + w_1(\epsilon^1) + w_2)/2$. In histories where $a_1 + w_1(\epsilon^1) < w_2$, in contrast, the borrowing constraint does bind and consumption in the second and third period equals $a_1 + w_1(\epsilon^1)$ and w_2, respectively. The value function thus equals

$$V_1(a_1 + w_1(\epsilon^1)) = \begin{cases} u(a_1 + w_1(\epsilon^1)) + u(w_2) & \text{if } w_1(\epsilon^1) < w_2 - a_1 \\ 2 \cdot u\left(\frac{a_1 + w_1(\epsilon^1) + w_2}{2}\right) & \text{if } w_1(\epsilon^1) \geq w_2 - a_1 \end{cases} .$$

Note that the derivative of the value function has a kink at the critical value $a_1 + w_1(\epsilon^1) = w_2$, below which consumption cannot be smoothed: $\lim_{\delta \downarrow 0} V_1''(w_2 - \delta) = u''(w_2)$ whereas $V_1''(w_2) = u''(w_2)/2$. That is, the derivative is convex around the critical level, independently of whether marginal utility is convex or not; all that is required for the convexity of V' is that preferences are strictly concave.

Consider now the effect of the potentially binding borrowing constraint at date $t = 1$ on saving in the initial period, a_1. While the household's program

$$\max_{a_1} \; u(w_0 - a_1) + \mathbb{E}_0[V_1(a_1 + w_1(\epsilon^1))]$$

yields the usual Euler equation, $u'(c_0) = \mathbb{E}_0[V_1'(a_1 + w_1(\epsilon^1))]$, the convexity of V' leads the household to save more at date $t = 0$ than if no risk of a binding borrowing constraint were present. The intuition mirrors the one for precautionary saving although it is the risk of a binding borrowing constraint in combination with strictly concave preferences—not convexity of marginal utility—which drives the result.

Buffer Stock Saving

Consider an impatient household in an environment with constant interest rates, $\beta R < 1$. Absent risk, this household would choose a declining consumption path. With risk, in contrast, the precautionary saving motive or the risk of a future binding borrowing constraint work in the opposite direction and encourage saving.

The net effect on saving depends on household wealth. When consumption is a concave function of *liquid wealth*—the sum of financial wealth and current labor income—then the motives that encourage saving weaken as the household becomes wealthier. Intuitively, when consumption is a concave function then the marginal propensity to

[1] Recall why a_2, the level of assets carried into date $t = 2$, is indexed by ϵ^1 rather than ϵ^2: This level is chosen at date $t = 1$ and as a consequence, it is the same in each history ϵ^2 subsequent to history ϵ^1. Formally, a_2 is measurable with respect to ϵ^1.

consume declines in liquid wealth; for a wealthier household, a given labor income risk therefore translates into lower consumption risk and thus, a weaker precautionary saving motive. The wealth dependent saving motive on the one hand and impatience on the other give rise to a target ratio of financial assets relative to average labor income. The household builds up a *buffer stock* of financial assets when its asset-to-income ratio is low and vice versa.

4.1.2 Complete Markets

Turning to *complete markets*, consider again the environment with two periods and two histories. In contrast to the incomplete market setting, the household now has access to two assets with linearly independent returns. For simplicity, we assume that these two assets are *Arrow securities* that is, securities that only pay off in one history each (we relax this assumption later). We denote by a_1^1 the savings in the first Arrow security which pays off if and only if $\epsilon^1 = h$, and we denote the history-dependent gross rate of return on this security by $R_1^1(\epsilon^1)$ with $R_1^1(l) = 0$. Similarly, a_1^2 denotes the savings in the second Arrow security which pays off if and only if $\epsilon^1 = l$, and its return is denoted $R_1^2(\epsilon^1)$ with $R_1^2(h) = 0$. The household's program reads

$$\begin{aligned} \max_{a_1^1, a_1^2, c_0, c_1(h), c_1(l)} \quad & u(c_0) + \beta(\eta(h)u(c_1(h)) + \eta(l)u(c_1(l))) \\ \text{s.t.} \quad & a_1^1 + a_1^2 = w_0 - c_0,\ c_1(\epsilon^1) = a_1^1 R_1^1(\epsilon^1) + a_1^2 R_1^2(\epsilon^1) + w_1(\epsilon^1). \end{aligned}$$

As in the incomplete-market setting, three dynamic budget constraints bind. In contrast to the incomplete-market setting, however, these three constraints can be combined into a single intertemporal budget constraint rather than separate ones for each history:

$$c_0 + \frac{c_1(h)}{R_1^1(h)} + \frac{c_1(l)}{R_1^2(l)} = w_0 + \frac{w_1(h)}{R_1^1(h)} + \frac{w_1(l)}{R_1^2(l)}.$$

The situation is akin to a static environment where the household can exchange all goods ($c_0, c_1(h)$, and $c_1(l)$) against each other—the household faces complete markets. In particular, and in contrast to the incomplete-market setting, the two assets do not only allow the household to shift purchasing power across time (that is, exchange c_0 against a *bundle* of $c_1(h)$ and $c_1(l)$) but also to specific nodes in the event tree. Equivalently, they allow to shift purchasing power across histories at date $t = 1$, by buying less of one Arrow security and more of the other. Since consumption in the two histories can be chosen independently of each other the household may achieve full insurance ($c_1(h) = c_1(l)$) although $w_1(h) \neq w_1(l)$, unlike in the incomplete-market case.

Figure 4.3 illustrates the complete-market setting. The dashed rectangle indicates the range of the single intertemporal budget constraint that connects the initial period and both histories in the second period. The arrows indicate the two margins of adjustment, corresponding to the choices of a_1^1 and a_1^2. Note that a simultaneous reduction

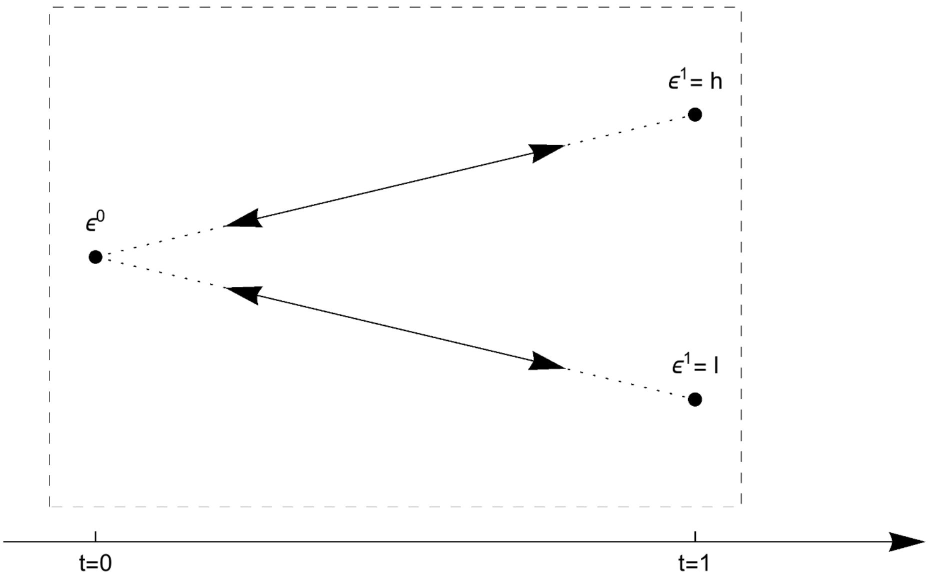

Figure 4.3: Complete markets: One intertemporal budget constraint and two adjustment margins.

of a_1^1 and increase of a_1^2 (or vice versa) allows to re-allocate purchasing power between the two histories.

The first-order conditions of the program with Arrow securities are given by the Euler equations

$$u'(c_0) = \beta R_1^1(h)\eta(h)u'(c_1(h)) \text{ and } u'(c_0) = \beta R_1^2(l)\eta(l)u'(c_1(l)).$$

These conditions do not contain an expectation operator because the choice of a_1^1 or a_1^2 affects second-period consumption only in one history each. If returns are actuarially fair that is, return differentials compensate for risk such that $R_1^1(h)/R_1^2(l) = \eta(l)/\eta(h)$, then it is optimal to smooth consumption across states, $c_1(h) = c_1(l)$. If, moreover, $\beta R_1^1(h)\eta(h) = 1$, then consumption is perfectly smoothed over time as well, unlike in the incomplete-market case.

Generalizations

Market completeness does not require the availability of Arrow securities. It only requires as many assets with linearly independent returns as there are branches of the event tree (which is guaranteed with a complete set of Arrow securities). To understand the independence requirement, consider an example with two histories and two assets with general return structure, $R_1^i(\epsilon^1) \geq 0, i = 1,2; \epsilon^1 = h,l$. (With Arrow securities, $R_1^1(l) = R_1^2(h) = 0$.) The dynamic budget constraints in the second period can be expressed as

$$\begin{bmatrix} c_1(h) - w_1(h) \\ c_1(l) - w_1(l) \end{bmatrix} = \begin{bmatrix} R_1^1(h) & R_1^2(h) \\ R_1^1(l) & R_1^2(l) \end{bmatrix} \begin{bmatrix} a_1^1 \\ a_1^2 \end{bmatrix}.$$

If the return vectors $R_1^1(\epsilon^1)$ and $R_1^2(\epsilon^1)$ are linearly independent then the matrix on the right-hand side has full rank and its determinant, $D = R_1^1(h)R_1^2(l) - R_1^1(l)R_1^2(h)$, differs from zero. The matrix then can be inverted and thus, the equation solved for a_1^1 and a_1^2. Substituting the resulting expressions into the first-period dynamic budget constraint yields the single intertemporal budget constraint

$$w_0 - c_0 + (w_1(h) - c_1(h))\frac{R_1^2(l) - R_1^1(l)}{D} + (w_1(l) - c_1(l))\frac{R_1^1(h) - R_1^2(h)}{D} = 0.$$

(When $R_1^1(l) = R_1^2(h) = 0$, this reduces to the constraint in the case with Arrow securities.)

The term $(R_1^2(l) - R_1^1(l))/D$ in the intertemporal budget constraint represents the price of second-period consumption in history h, expressed in terms of first-period consumption. To see this, note that purchasing ϕ units of the first asset and $-\phi R_1^1(l)/R_1^2(l)$ units of the second yields an absolute return of $\phi(R_1^1(h) - R_1^2(h)R_1^1(l)/R_1^2(l))$ in history h and zero in history l. To secure one additional unit of consumption in history h, the household thus must acquire $\phi = (R_1^1(h) - R_1^2(h)R_1^1(l)/R_1^2(l))^{-1} = R_1^2(l)/D$ units of the first asset and $-R_1^1(l)/D$ units of the second, at a cost of $(R_1^2(l) - R_1^1(l))/D$.[2] Similarly, $(R_1^1(h) - R_1^2(h))/D$ represents the price of consumption in history l.

In an interior equilibrium, the Euler equations now read

$$\begin{aligned} u'(c_0) &= \beta(R_1^1(h)\eta(h)u'(c_1(h)) + R_1^1(l)\eta(l)u'(c_1(l))), \\ u'(c_0) &= \beta(R_1^2(h)\eta(h)u'(c_1(h)) + R_1^2(l)\eta(l)u'(c_1(l))). \end{aligned}$$

Linear combinations of these equations recover the Euler equations for the Arrow securities. For example, multiplying the first equation by $R_1^2(l)$ and the second by $-R_1^1(l)$ and summing yields

$$u'(c_0) = \beta\frac{D}{R_1^2(l) - R_1^1(l)}\eta(h)u'(c_1(h)).$$

Market completeness does not require all date- and history-contingent goods to be traded in the initial period (either by means of Arrow securities or combinations of assets with linearly independent returns). It suffices when at each node of the event tree, there are securities to shift purchasing power across adjacent branches of the event tree and all goods can be traded on spot markets.

[2] When $R_1^1(l) = R_1^2(l)$ but $D \neq 0$ then the return on one asset strictly dominates the return on the other: In history l both assets generate the same return, but in history h one generates a strictly higher return than the other (if $D > 0$ the first asset returns more, if $D < 0$ the second does). Buying the asset with the strictly higher return and selling the one with the lower return allows to increase consumption in history h without having to give up consumption in the initial period or in history l; the price of consumption in state h therefore equals zero. See also section 5.3.

To see this, consider an economy with three periods, $t = 0, 1, 2$; S states of nature in both the second and the third period and thus, S^2 histories at date $t = 2$; G goods at each node of the event tree in the second and third period; and one good at date $t = 0$. At each node, one good serves as numeraire. A complete set of Arrow securities includes $(S + S^2)G$ securities, namely SG for the delivery of history-contingent goods at date $t = 1$ and S^2G for delivery at date $t = 2$.[3] Note that the returns on the Arrow securities implicitly define relative prices between the goods in each node, and between the numeraire goods in different nodes.

Consider next an alternative market structure where only S assets with linearly independent returns (e.g., S Arrow securities) are traded at date $t = 0$. Once uncertainty is resolved at date $t = 1$, the numeraire good is traded against the other goods on spot markets and in addition, again, S assets with linearly independent returns are traded. When uncertainty is resolved at date $t = 2$, all goods again are traded on spot markets. This alternative market structure only uses $S + G + S + G$ markets but it provides the same trading possibilities as the complete set of Arrow securities. It also generates the same budget set when the spot market prices and asset returns correspond to the implied relative prices in the environment with Arrow securities.

4.1.3 General Case

We have seen that the household faces a single intertemporal budget constraint when markets are complete and multiple constraints when they are incomplete. To clarify this point under more general conditions consider a two-period setup with a finite number of histories and a finite number of assets indexed by i. Markets may be complete or incomplete. The household's program reads

$$\begin{aligned} \max_{c_0, \{a_1^i\}_i, \{c_1(\epsilon^1)\}_{\epsilon^1}} \quad & u(c_0) + \beta \mathbb{E}_0 \left[u(c_1(\epsilon^1)) \right] \\ \text{s.t.} \quad & \sum_i a_1^i = w_0 - c_0, \ c_1(\epsilon^1) = \sum_i a_1^i R_1^i(\epsilon^1) + w_1(\epsilon^1). \end{aligned}$$

For each asset i that the household purchases or sells the corresponding Euler equation $u'(c_0) = \beta \mathbb{E}_0[u'(c_1(\epsilon^1))R_1^i(\epsilon^1)]$ holds. Expressed differently, $1 = \mathbb{E}_0[m_1(\epsilon^1)R_1^i(\epsilon^1)]$ where $m_1(\epsilon^1) \equiv \beta u'(c_1(\epsilon^1))/u'(c_0)$ denotes the household's *stochastic discount factor*, namely the marginal rate of substitution normalized by the conditional probability of history ϵ^1. Note that

$$\sum_i a_1^i = \mathbb{E}_0 \left[m_1(\epsilon^1) \sum_i a_1^i R_1^i(\epsilon^1) \right]$$

because the household either is invested in asset i, in which case $a_1^i = \mathbb{E}_0[m_1(\epsilon^1)a_1^i R_1^i(\epsilon^1)]$, or it is not invested in which case $a_1^i = 0$.

[3]Note that we specify Arrow securities to be time, history, and good specific.

Multiplying the dynamic budget constraints at date $t = 1$ by $m_1(\epsilon^1)$ and taking expectations yields

$$\mathbb{E}_0[m_1(\epsilon^1)c_1(\epsilon^1)] = \mathbb{E}_0\left[m_1(\epsilon^1)\sum_i a_1^i R_1^i(\epsilon^1)\right] + \mathbb{E}_0[m_1(\epsilon^1)w_1(\epsilon^1)].$$

Adding the dynamic budget constraint at date $t = 0$, we arrive at the equilibrium condition

$$c_0 + \mathbb{E}_0[m_1(\epsilon^1)c_1(\epsilon^1)] = w_0 + \mathbb{E}_0[m_1(\epsilon^1)w_1(\epsilon^1)]. \quad (4.1)$$

Condition (4.1) holds independently of whether markets are complete or incomplete. It integrates all dynamic budget constraints and uses the household's Euler equations to express prices in terms of stochastic discount factors. When markets are complete, the single intertemporal condition (4.1) fully represents the household's budgetary restrictions conditional on the equality of marginal rates of substitution and prices. When markets are incomplete, in contrast, condition (4.1) represents the budgetary restrictions only partially because it does not account for the market incompleteness which prevents purchasing power to be shifted across time and histories in arbitrary ways. That is, market incompleteness imposes constraints in addition to condition (4.1).

As an example, recall the saving problem with incomplete markets (and a risk-free return) considered earlier. The history-contingent intertemporal budget constraint and the Euler equation are given by

$$w_0 - c_0 + \frac{w_1(\epsilon^1) - c_1(\epsilon^1)}{R_1} = 0, \; \frac{1}{R_1} = \mathbb{E}_0\left[m_1(\epsilon^1)\right].$$

Summing the intertemporal budget constraints for the two histories, weighted by the respective probabilities and stochastic discount factors, yields condition (4.1) once the Euler equation is imposed. But condition (4.1) does not imply the two intertemporal budget constraints.

4.2 Risk Sharing

4.2.1 Borch's Rule

Consider an economy with heterogeneous groups of representative households who buy and sell assets with contingent returns. In equilibrium at date t,

$$1 = \mathbb{E}_t[m_{t+1}^h(\epsilon^{t+1})R_{t+1}^i(\epsilon^{t+1})]$$

for all assets i and all (representative) households h where m_{t+1}^h denotes h's stochastic discount factor. For households from two different groups, l and n say, it follows that

$$\mathbb{E}_t[(m_{t+1}^l(\epsilon^{t+1}) - m_{t+1}^n(\epsilon^{t+1}))R_{t+1}^i(\epsilon^{t+1})] = 0.$$

Each asset with a linearly independent return vector imposes one constraint of this type. When markets are complete the combined restrictions imply that $m^l_{t+1}(\epsilon^{t+1}) = m^n_{t+1}(\epsilon^{t+1})$ in each history. This is most easily seen in the case where the assets include a complete set of Arrow securities.

When markets are complete such that $m^l_{t+1}(\epsilon^{t+1}) = m^n_{t+1}(\epsilon^{t+1})$ and both households share the same utility function then

$$\frac{u'(c^n_t(\epsilon^t))}{u'(c^l_t(\epsilon^t))} = \frac{u'(c^n_{t+1}(\epsilon^{t+1}))}{u'(c^l_{t+1}(\epsilon^{t+1}))}. \tag{4.2}$$

Condition (4.2), which is referred to as *Borch's rule*, states that households *share risk*—whenever marginal utility of one household is high or low the same holds true for the other. The ratio of marginal utilities reflects differences in wealth. When household preferences are homothetic then condition (4.2) simplifies to

$$\frac{c^n_t(\epsilon^t)}{c^l_t(\epsilon^t)} = \frac{c^n_{t+1}(\epsilon^{t+1})}{c^l_{t+1}(\epsilon^{t+1})}.$$

Risk sharing is Pareto optimal. To see this, consider the problem of maximizing the welfare of household l subject to the other households attaining given levels of welfare, $\{\bar{U}^h\}_h$. The Lagrangian reads

$$\begin{aligned} \mathcal{L} &= \mathbb{E}_0\left[\sum_{t=0}^{\infty}(\beta^l)^t u(c^l_t(\epsilon^t))\right] + \sum_{h\neq l}\lambda^h \mathbb{E}_0\left[\sum_{t=0}^{\infty}(\beta^h)^t u(c^h_t(\epsilon^t)) - \bar{U}^h\right] \\ &\quad + \sum_{t,\epsilon^t}\mu_t(\epsilon^t)\left\{\ldots - c^l_t(\epsilon^t) - \sum_{h\neq l} c^h_t(\epsilon^t) + \ldots\right\}, \end{aligned}$$

where we assume that the number of households is the same in each group; λ^h denotes the multiplier associated with the reservation utility requirement for household h; and $\mu_t(\epsilon^t)$ denotes the multiplier associated with the resource constraint. We do not need to be specific about the production side of the economy, thus the dots. Differentiating with respect to consumption yields

$$u'(c^l_t(\epsilon^t)) = \lambda^h u'(c^h_t(\epsilon^t))$$

which implies the risk sharing condition (4.2).

4.2.2 Aggregate and Idiosyncratic Risk

Suppose that the endowment of household h, $w^h_t(\epsilon^t)$, has an *aggregate* and an *idiosyncratic* (that is, household specific) component. The former, $w_t(\epsilon^t)$, is the same across all groups while the latter, $\iota^h_t(\epsilon^t)$, differs across groups; the sum of the idiosyncratic components across households equals zero in all histories. Markets are complete and in equilibrium, households thus share risk.

Let φ^h denote wealth of household h relative to average wealth. With identical and homothetic preferences condition (4.2) then implies $c_t^h(\epsilon^t) = \varphi^h c_t(\epsilon^t)$ where $c_t(\epsilon^t)$ denotes average consumption. Using the resource constraint, $w_t(\epsilon^t) = c_t(\epsilon^t)$, this yields

$$c_t^h(\epsilon^t) = \varphi^h w_t(\epsilon^t).$$

Note that consumption of all households is proportional to the average endowment in the economy. In other words, with complete markets consumption of households only reflects aggregate shocks and idiosyncratic risk is fully insured.

4.3 Uninsurable Labor Income Risk and Capital Accumulation

In stark contrast to the environment with risk sharing we now consider a setting where insurance is ruled out. Specifically, we study the consequences of *uninsurable idiosyncratic labor income risk* for capital accumulation.

4.3.1 Economy

The structure of the economy is the same as in the representative agent model of section 3.1, except for one difference: The time endowment of each household is random. Formally, there is a continuum of measure one of infinitely lived households, indexed by $h \in [0, 1]$. The time endowment of household h at date t is given by $1 + \iota_t^h$. It is strictly positive, bounded, and i.i.d. across households with minimum value $1 + \underline{\iota}$ and mean $\mathbb{E}_t[1 + \iota_{t+1}^h] = 1$. As a consequence, aggregate labor supply equals unity at all times. We assume that the time endowment follows a *Markov process* that is, the probability that the endowment takes a specific value at date t only depends on the realized value at date $t - 1$.

Households have access to a risk-free asset with gross interest rate R_t. Net financial assets of households correspond to the economy's capital stock, k_t. Firms rent labor at the competitive wage w_t per unit of time, and capital at the competitive rate r_t. Capital depreciates at rate δ and thus, $R_t = 1 + r_t - \delta$.

We consider a *stationary equilibrium*: While the time endowment and assets of an individual household change from period to period, the joint distribution of time endowments and assets across the population is time invariant. Accordingly, the aggregate capital stock and thus, interest rates and wages are time invariant as well.

4.3.2 Households

Household h maximizes $\mathbb{E}_0[\sum_{t=0}^{\infty} \beta^t u(c_t^h(\epsilon^t))]$. Its dynamic budget constraint is given by

$$a_{t+1}^h(\epsilon^t) = a_t^h(\epsilon^{t-1})R + w(1 + \iota_t^h(\epsilon^t)) - c_t^h(\epsilon^t).$$

Consumption must be non-negative. As debt must be serviced under all circumstances this implies a *natural borrowing limit* equal to the market value of future labor income in the worst possible history,

$$a^i_{t+1}(\epsilon^t) \geq \underline{a} \equiv -w(1+\underline{\iota})\frac{1}{R-1}.$$

In addition, a tighter borrowing constraint may bind. For example, if households are excluded from borrowing at all then the natural borrowing limit is replaced by the restriction $a^h_{t+1}(\epsilon^t) \geq 0$.

At date t, history ϵ^t the state variables in the household's program are $(a^h_t(\epsilon^{t-1}), \iota^h_t(\epsilon^t))$ as well as the constant wage and interest rate, and the control variables are $(a^h_{t+1}(\epsilon^t), c^h_t(\epsilon^t))$. Let $(a^h_\circ, \iota^h_\circ)$ denote household assets and the time endowment in the current period, and let (a^h_+, ι^h_+) denote those objects in the subsequent period. Since the program is time autonomous the Bellman equation for household h reads

$$V(a^h_\circ, \iota^h_\circ; w, R) = \max_{a^h_+} u(a^h_\circ R + w(1+\iota^h_\circ) - a^h_+) + \beta \mathbb{E}\left[V(a^h_+, \iota^h_+; w, R)|\iota^h_\circ\right]$$

subject to the borrowing constraint; the expectation is conditional on $\iota^h_\circ$ because the current time endowment may contain information about the probability distribution of next period's endowment (the Markov assumption implies that $\iota^h_\circ$ contains all such information).

If ι risk were absent (or equivalently, if markets were complete and households insured each other), the household would face a risk-free, constant labor income stream. With $\beta R = 1$, its optimal consumption would be time invariant and equal to $a^h_\circ(R-1) + w$. With risk, in contrast, optimal consumption cannot be constant and finite. If it were, its value would have to be consistent with the worst case scenario of minimum time endowments forever after. But after a more favorable time endowment realization the household could increase consumption and this implies a contradiction.

Suppose that $\beta R \geq 1$. The household's first-order and envelope conditions then yield

$$V_a(a^h_\circ, \iota^h_\circ; w, R) = Ru'(c^h_\circ) \geq \mathbb{E}\left[V_a(a^h_+, \iota^h_+; w, R)|\iota^h_\circ\right] = R\mathbb{E}\left[u'(c^h_+)|\iota^h_\circ\right].$$

With $u' > 0, u'' < 0$, optimal consumption thus stochastically converges to infinity as the household accumulates more and more assets to *self insure* against low future time endowment realizations. Formally, the Euler equation $u'(c^h_\circ) \geq \mathbb{E}\left[u'(c^h_+)|\iota^h_\circ\right]$ implies that marginal utility follows a submartingale, which converges; and strict concavity of the utility function implies that marginal utility only converges if consumption converges. Since an income shock translates into a change of consumption unless asset holdings are infinite, convergence of marginal utility therefore requires that asset holdings converge to infinity.

We conclude that for households to accumulate a finite level of assets the interest rate must satisfy $\beta R < 1$.

4.3.3 General Equilibrium

The stationary equilibrium in the hypothetical economy without risk (or with insurance) would satisfy $R = \beta^{-1} = 1 + r - \delta$, $f_K(k, 1) = r$, and $f_L(k, 1) = w$. Aggregate consumption would equal $k(R - 1) + w$.

In the stationary equilibrium in the economy with risk, in contrast, R must be strictly smaller than β^{-1} to clear the market for capital; otherwise the supply of capital would grow without bound while firms' demand would be bounded. Households accumulate assets when their time endowment is high and run them down when it is low. The capital stock in the economy is constant and since $R < \beta^{-1}$, it is strictly larger than in the economy without risk, and so are wages. Although the risk is purely idiosyncratic and washes out in the aggregate, self insurance gives rise to a higher capital stock.

Suppose for simplicity that the time endowment can assume m possible values and the asset holdings of a household n values; both m and n are finite. The $m \times m$ transition matrix Π^ι whose rows sum to unity contains the transition probabilities of the time endowment; the (i, j) element of Π^ι gives the probability that ι_+ takes the j-th of the m possible values conditional on ι taking the i-th such value.

The state (a^h, ι^h) of household h then can take mn values. Together with the transition matrix Π^ι, the decision rules of households define a transition matrix for this state, Π say which is of size $mn \times mn$. Let $\Pi^\top$ denote the transpose of Π, and let d of size $mn \times 1$ denote the probability distribution of households over the possible states; the elements of d sum to one. Note that conditional on $d_\circ$, the probability distribution in the subsequent period is given by $d_+ = \Pi^\top d_\circ$. A *stationary distribution* therefore satisfies $d = \Pi^\top d$; it is the normalized eigenvector associated with the unit eigenvalue of $\Pi^\top$.

4.4 Bibliographic Notes

von Neumann and Morgenstern (1944) introduce expected utility. Modigliani and Brumberg (1954) discuss the precautionary savings motive and Friedman (1957, p. 16) discusses saving as a "reserve for emergencies." Phelps (1962) and Levhari and Srinivasan (1969) study dynamic optimization problems with risky returns on saving. Leland (1968) relates the precautionary saving motive to the convexity of marginal utility, and Sandmo (1970) analyzes the differences between return and labor income risk. The analysis of the saving problem with quadratic utility is due to Hall (1978). Kimball (1990) defines prudence as the sensitivity of an optimal choice (here saving) to risk. Zeldes (1989*a*; 1989*b*) analyzes how the risk of a binding borrowing constraint and precautionary motives affect the consumption function. Deaton (1991) analyzes buffer stock saving in a model with borrowing constraints, precautionary motive, and impatience, and Carroll (1997) shows that impatient households with a precautionary motive target a wealth-to-permanent-income ratio. Arrow (1953; 1964) and Radner (1972) analyze equilibrium with sequential trading.

Borch (1962) derives the risk sharing condition.

Aiyagari (1994) analyzes idiosyncratic risk and capital accumulation in stationary equilibrium. Chamberlain and Wilson (2000) prove that consumption grows without bound when $\beta R \geq 1$.

Related Topics Cass and Shell (1983) analyze the efficiency implications of risk and *limited participation* in an overlapping generations economy. They show that even with complete markets and a finite horizon, extrinsic risk (sunspots) can cause a Pareto inefficient allocation because the yet unborn do not participate in markets.

Aiyagari's (1994) model introduces capital into frameworks developed by Bewley (1977; 1980; 1986) and Huggett (1993) (see sections 5.6 and 9.4). Angeletos (2007) analyzes the implications of *uninsurable idiosyncratic capital income risk*. As in Aiyagari's (1994) model the interest rate is lower than with complete markets but unlike in that model, the capital stock need not exceed its complete markets level, due to a risk premium which drives a wedge between the interest rate and the marginal product of capital.

Werning (2015) studies the effect of market incompleteness on aggregate consumption.

Deaton (1992) covers consumption. Laffont (1989) and Gollier (2001) cover models of risk, time, and information.

See section 8.3 for models with financial frictions.

Chapter 5

Asset Returns and Asset Prices

If a household invests in multiple assets it is indifferent between the investments at the margin. We derive the implications of this indifference and of market clearing for asset returns and prices. Throughout we assume that households maximize expected utility.

5.1 Euler Equation

Consider the equilibrium in an economy with two periods and risk. Markets may be complete or incomplete. We saw earlier (see subsection 4.1.3) that for each asset i and each household h that purchases or sells the asset, an Euler equation

$$u'(c_0^h) = \beta \mathbb{E}_0[u'(c_1^h(\epsilon^1))R_1^i(\epsilon^1)] \text{ or } 1 = \mathbb{E}_0[m_1^h(\epsilon^1)R_1^i(\epsilon^1)]$$

holds where $m_1^h(\epsilon^1) \equiv \beta u'(c_1^h(\epsilon^1))/u'(c_0^h)$ denotes household h's stochastic discount factor.

This has two important implications. First, when asset i is held by different households, l and n say, then the stochastic discount factors of these households satisfy $\mathbb{E}_0[m_1^l(\epsilon^1)R_1^i(\epsilon^1)] = \mathbb{E}_0[m_1^n(\epsilon^1)R_1^i(\epsilon^1)]$ that is, the return weighted average stochastic discount factors coincide. When l and n hold multiple assets this imposes multiple cross-household restrictions. When markets are complete, the cross-household restrictions imply that $m_1^l(\epsilon^1) = m_1^n(\epsilon^1)$ for each history (see subsection 4.2.1).

The second implication concerns return differentials across assets to which we turn next.

5.2 Excess Returns

5.2.1 C-CAPM

When a household with stochastic discount factors $\{m_1(\epsilon^1)\}_{\epsilon^1}$ purchases or sells multiple assets, j and k say, then the rates of returns on these assets satisfy $\mathbb{E}_0[m_1(\epsilon^1)R_1^j(\epsilon^1)] = \mathbb{E}_0[m_1(\epsilon^1)R_1^k(\epsilon^1)]$ that is, the stochastic-discount-factor weighted average returns on the assets coincide. Expressing the equality as $\mathbb{E}_0[m_1(\epsilon^1)(R_1^j(\epsilon^1) - R_1^k(\epsilon^1))] = 0$ and using the definition of covariance yields

$$\mathbb{E}_0[m_1(\epsilon^1)]\mathbb{E}_0[R_1^j(\epsilon^1) - R_1^k(\epsilon^1)] + \mathbb{C}\text{ov}_0[m_1(\epsilon^1), R_1^j(\epsilon^1) - R_1^k(\epsilon^1)] = 0.$$

Equilibrium therefore imposes restrictions on the *expected returns* and *return covariances* of assets. The latter reflect how strongly asset returns covary with the stochastic discount factor.

Suppose that asset f is risk-free, $R_1^f(\epsilon^1) = R_1^f$, and the household holds the risk-free asset such that $1 = \mathbb{E}_0[m_1(\epsilon^1)]R_1^f$. The *excess return* of asset i that is, the expected rate of return net of the risk-free return then satisfies

$$\mathbb{E}_0[R_1^i(\epsilon^1)] - R_1^f = -\frac{\mathbb{C}\text{ov}_0[m_1(\epsilon^1), R_1^i(\epsilon^1)]}{\mathbb{E}_0[m_1(\epsilon^1)]} = -\mathbb{C}\text{ov}_0[m_1(\epsilon^1), R_1^i(\epsilon^1)]R_1^f.$$

According to this *consumption capital asset pricing model* (C-CAPM) result, the excess return is proportional to the covariance between the asset's return and the stochastic discount factor. Note that the excess return compensates for covariation of the asset return with marginal utility, not for return volatility per se.

Since β and $u'(c_0)$ in $m_1(\epsilon^1)$ are constants, the sign of the excess return depends on the sign of the covariance between $R_1^i(\epsilon^1)$ and $u'(c_1(\epsilon^1))$. The asset pays zero excess return if this covariance is zero, for example because utility is linear (risk neutrality) or consumption is deterministic (full insurance). If the asset return covaries negatively with the stochastic discount factor and thus (if utility is strictly concave) positively with $c_1(\epsilon^1)$, then the excess return is positive. Intuitively, the asset is a bad *hedge* in this case; it tends to pay more when the marginal benefit from additional resources is small. To induce the household to nevertheless hold the asset its return must be high. If the asset return covaries negatively with $c_1(\epsilon^1)$, in contrast, then the excess return is negative; when the asset is a good hedge it need not pay a high average return.

The C-CAPM implies a *mean-variance frontier* that bounds the absolute value of an asset's excess return given the standard deviation of its return:

$$\begin{aligned} & \mathbb{E}_0[R_1^i(\epsilon^1)] - R_1^f = -\frac{\mathbb{C}\text{ov}_0[m_1(\epsilon^1), R_1^i(\epsilon^1)]}{\mathbb{E}_0[m_1(\epsilon^1)]} \\ \Rightarrow \quad & |\mathbb{E}_0[R_1^i(\epsilon^1)] - R_1^f| \leq \frac{\mathbb{S}\text{td}_0[m_1(\epsilon^1)]\mathbb{S}\text{td}_0[R_1^i(\epsilon^1)]}{\mathbb{E}_0[m_1(\epsilon^1)]}. \end{aligned}$$

Here, we use the fact that the covariance equals the product of the standard deviations and the correlation coefficient, which lies between minus and plus one.

5.2.2 CAPM

The C-CAPM establishes a linear relation between the equilibrium excess return on an asset and the covariance between the asset return and the stochastic discount factor.

The *capital asset pricing model* (CAPM), which precedes the C-CAPM, similarly establishes such a linear relation; but in the case of the CAPM it is the covariance between the asset return and the return on the *market portfolio* encompassing all risky assets which enters the relation.

The CAPM follows from the C-CAPM under the assumption that consumption is a linear function of the rate of return on the market portfolio—the *market return* $R_1^m(\epsilon^1)$—and the marginal utility function is accurately approximated to the first order. The stochastic discount factor, $m_1(\epsilon^1)$, then is a linear function of $R_1^m(\epsilon^1)$,

$$\mathbb{E}_0[R_1^i(\epsilon^1)] - R_1^f = \mathbb{C}\text{ov}_0\left[R_1^m(\epsilon^1), R_1^i(\epsilon^1)\right]\phi R_1^f,$$

where ϕ denotes a factor of proportionality. In particular, the excess return on the market portfolio satisfies

$$\mathbb{E}_0[R_1^m(\epsilon^1)] - R_1^f = \mathbb{C}\text{ov}_0[R_1^m(\epsilon^1), R_1^m(\epsilon^1)]\phi R_1^f.$$

Solving for ϕ and substituting into the previous equation, it follows that

$$\mathbb{E}_0[R_1^i(\epsilon^1)] - R_1^f = \frac{\mathbb{C}\text{ov}_0[R_1^m(\epsilon^1), R_1^i(\epsilon^1)]}{\mathbb{V}\text{ar}_0[R_1^m(\epsilon^1)]}(\mathbb{E}_0[R_1^m(\epsilon^1)] - R_1^f).$$

The ratio on the right-hand side of the last equality represents asset i's *beta*, the normalized covariance between the asset return and the market return. (Formally, beta equals the projection of $R_1^i(\epsilon^1)$ on $R_1^m(\epsilon^1)$.) According to the CAPM, the excess return equals the product of the asset's beta and the market's excess return.

Originally, the CAPM was derived under the assumption that a representative household values the mean return on its portfolio (positively) as well as the return variance (negatively). The optimal *portfolio choice* then implies a linear relation between an asset's excess return and the covariance between the asset and portfolio returns. Moreover, market clearing requires that the household's portfolio choice corresponds to the market portfolio and thus, the portfolio return to the market return.

Formally, let e denote the $n \times 1$ vector of expected returns on n risky assets; and V the $n \times n$ variance-covariance matrix of the returns. The household chooses the portfolio shares invested in the risky assets, represented by the $n \times 1$ vector x, and maximizes the expected portfolio return minus γ times the portfolio variance, where γ reflects risk aversion. Letting a T superscript denote transposition and $\bar{x}$ the portfolio share $1 - \sum_{i=1}^n x_i$ invested in the risk-free asset, the household's problem reads

$$\max_x \ x^T e + (1 - \bar{x})R^f - \gamma x^T V x$$

and yields the first-order condition

$$e^T - R^f = 2\gamma\, x^T V.$$

Post multiplying the last condition by x and combining the result with the first-order condition yields

$$e^T - R^f = \frac{x^T V}{x^T V x}[(e^T - R^f)x].$$

Market clearing implies that the shares chosen by the household, x, correspond to the shares of the risky assets in the market portfolio (the former are $1 - \bar{x}$ times the latter). The ratio on the right-hand side of the equation thus corresponds to the vector of betas; and the expression in brackets corresponds to the excess return on the market portfolio. The difference on the left-hand side represents the vector of excess returns.

5.3 Asset Prices

To derive the implications of the C-CAPM for *asset prices*, we use the definition of a return: The gross rate of return between date t and $t+1$, $R^i_{t+1}(\epsilon^{t+1})$, equals the payoff at date $t+1$ relative to the asset price at date t, $p^i_t(\epsilon^t)$; and the payoff consists of the asset price, $p^i_{t+1}(\epsilon^{t+1})$, and the dividend, $d^i_{t+1}(\epsilon^{t+1})$:

$$R^i_{t+1}(\epsilon^{t+1}) \equiv \frac{p^i_{t+1}(\epsilon^{t+1}) + d^i_{t+1}(\epsilon^{t+1})}{p^i_t(\epsilon^t)}. \tag{5.1}$$

We can therefore rewrite the equilibrium condition $1 = \mathbb{E}_t[m_{t+1}(\epsilon^{t+1})R^i_{t+1}(\epsilon^{t+1})]$ as

$$p^i_t(\epsilon^t) = \mathbb{E}_t[m_{t+1}(\epsilon^{t+1})(p^i_{t+1}(\epsilon^{t+1}) + d^i_{t+1}(\epsilon^{t+1}))].$$

Conditional on $\{m_{t+1}(\epsilon^{t+1})\}_{\epsilon^{t+1}}$ and a probability distribution over histories, any asset with specified payoffs thus can be priced by computing the expectation of the stochastic discount factor times the payoff.

Financial economists refer to $\{m_{t+1}(\epsilon^{t+1})\}_{\epsilon^{t+1}}$ as the *asset pricing kernel* or stochastic discount factor. Rather than relating the pricing kernel to the marginal rate of substitution and thus, consumption they often take it as given. The *law of one price* in financial economics states that portfolios with identical payoffs have the same price; the law holds as long as every portfolio that pays off zero in each history, has a price of zero. When markets are complete the law of one price implies a unique asset pricing kernel and unique prices for all Arrow securities, referred to as *state prices*. An *arbitrage* is a portfolio with a strictly negative price that pays off a non-negative amount in every history, or a portfolio with a zero price that pays off a non-negative amount in every history and a strictly positive amount in some histories. Absence of arbitrage is equivalent to strictly positive state prices. If utility functions are strictly increasing equilibrium therefore requires the absence of arbitrage. In the presence of portfolio restrictions (for example short-sale constraints) the law of one price may not hold and equilibrium does not require the absence of arbitrage.

5.3.1 Fundamental Value

With multiple periods, iterating the pricing equation forward T times yields

$$\begin{aligned} p_0^i &= \mathbb{E}_0\left[m_1(\epsilon^1)\left(d_1^i(\epsilon^1)+\mathbb{E}_1\left[m_2(\epsilon^2)\left(d_2^i(\epsilon^2)+\ldots+\mathbb{E}_{T-1}\left[m_T(\epsilon^T)d_T^i(\epsilon^T)\right]\right)\right]\right)\right] \\ &\quad +\mathbb{E}_0\left[m_1(\epsilon^1)\mathbb{E}_1\left[m_2(\epsilon^2)\ldots\mathbb{E}_{T-1}\left[m_T(\epsilon^T)p_T^i(\epsilon^T)\right]\right]\right] \\ &= \mathbb{E}_0\left[\sum_{s=1}^{T}(m_1(\epsilon^1)\cdots m_s(\epsilon^s))d_s^i(\epsilon^s)\right]+\mathbb{E}_0\left[(m_1(\epsilon^1)\cdots m_T(\epsilon^T))p_T^i(\epsilon^T)\right] \\ &= \mathbb{E}_0\left[\sum_{s=1}^{T}\beta^s\frac{u'(c_s(\epsilon^s))}{u'(c_0)}d_s^i(\epsilon^s)\right]+\mathbb{E}_0\left[\beta^T\frac{u'(c_T(\epsilon^T))}{u'(c_0)}p_T^i(\epsilon^T)\right], \end{aligned}$$

where we use the law of iterated expectations and

$$m_1(\epsilon^1)\cdots m_T(\epsilon^T)=\beta\frac{u'(c_1(\epsilon^1))}{u'(c_0)}\cdots\beta\frac{u'(c_T(\epsilon^T))}{u'(c_{T-1}(\epsilon^{T-1}))}=\beta^T\frac{u'(c_T(\epsilon^T))}{u'(c_0)}.$$

The asset price has two components: The expected present discounted value of the dividend stream until date $t=T$; and the expected present discounted value of the price at this date. If T is the final period such that $p_T^i(\epsilon^T)=0$ then the former component is the asset's *fundamental value*.

If the asset has an infinite maturity then its price satisfies

$$p_0^i=\lim_{T\to\infty}\mathbb{E}_0\left[\sum_{s=1}^{T}\beta^s\frac{u'(c_s(\epsilon^s))}{u'(c_0)}d_s^i(\epsilon^s)\right]+\lim_{T\to\infty}\mathbb{E}_0\left[\beta^T\frac{u'(c_T(\epsilon^T))}{u'(c_0)}p_T^i(\epsilon^T)\right].$$

Again, it has two components, a fundamental value and a *bubble* component (the rightmost term). Whether p_0^i exceeds the fundamental value depends on whether the bubble component is strictly positive and thus, whether $p_T^i(\epsilon^T)$ grows more quickly than $\beta^T u'(c_T(\epsilon^T))/u'(c_0)$ shrinks as $T\to\infty$. We turn next to the question whether this is possible.

5.3.2 Bubble

For simplicity, suppose that the utility function is linear and dividends are constant such that $m_t(\epsilon^t)=\beta$, $d_t^i(\epsilon^t)=d^i$, and

$$p_0^i=\lim_{T\to\infty}\sum_{s=1}^{T}\beta^s d^i+\lim_{T\to\infty}\beta^T\mathbb{E}_0[p_T^i(\epsilon^T)].$$

One solution to this equation is a constant price equal to the fundamental value, $p_t^i=d^i\beta/(1-\beta)$; the bubble component equals zero in this case. Another candidate solution is $p_t^i=d^i\beta/(1-\beta)+\text{bubble}_t^i$ where $\{\text{bubble}_t^i\}_{t\geq 0}$ is a strictly positive sequence satisfying $\text{bubble}_t^i = \beta\, \text{bubble}_{t+1}^i$; that is, the bubble grows at the rate of interest.[1] This candidate solution satisfies the asset pricing equation because

$$p_t^i = d^i \frac{\beta}{1-\beta} + \text{bubble}_t^i = \beta d^i + \beta d^i \frac{\beta}{1-\beta} + \beta\, \text{bubble}_{t+1}^i = \beta(d^i + p_{t+1}^i).$$

To check whether the candidate solution with a bubble component is consistent with rational expectations, suppose first that the number of potential investors is finite. In this case it is impossible that all households purchasing the asset at a bubbly price expect somebody else to purchase it at an even higher bubbly price in the future. A bubbly price therefore is inconsistent with common knowledge in a rational expectations equilibrium.

Suppose next that new potential investors enter the economy as time progresses. A household may then purchase the asset at a bubbly price expecting to resell it to subsequent investors with similar expectations. When the interest rate strictly exceeds the economy's growth rate then such expectation formation cannot be rational; a bubble growing at the rate of interest would eventually outgrow the economy and newcomers would not be able to purchase the bubble any longer. But when the interest rate falls short of the growth rate, then a bubble may be sustained in rational expectations equilibrium.

Recall that the growth rate in an inefficient overlapping generations economy exceeds the interest rate. Such an environment therefore admits bubbles. In fact, a bubble can play exactly the same role as a Pareto improving inter generational transfer scheme (see subsection 3.2.7): When an initial old cohort creates a pure bubble and old households in each period sell the bubble to young ones the latter transfer resources to the former; this absorbs saving of the young and reduces or eliminates capital over accumulation. A pure bubble of this type can be interpreted as money (see section 9.2). While equilibrium imposes restrictions on the growth rate of the bubble it is consistent with infinitely many bubble sizes. That is, the equilibrium allocation with a bubble is *indeterminate*.

5.4 Term Structure of Interest Rates

The price of a risk-free one period bond that pays off unity equals

$$p_t^{f1}(\epsilon^t) = \mathbb{E}_t \left[m_{t+1}(\epsilon^{t+1})\, 1 \right],$$

and the risk-free one period gross interest rate, $R_{t+1}^{f1}(\epsilon^t)$, equals the inverse of the bond price (from condition (5.1)). (Note that the risk-free interest rate is indexed by ϵ^t because the return is the same across all histories ϵ^{t+1} subsequent to ϵ^t.) More generally, a risk-free s period bond that pays off unity is priced at

$$p_t^{fs}(\epsilon^t) = \mathbb{E}_t \left[m_{t+1}(\epsilon^{t+1}) \cdots m_{t+s}(\epsilon^{t+s})\, 1 \right]$$

[1]Still other candidate solutions involve stochastic bubbles that grow at a faster rate but collapse stochastically.

and the risk-free s period gross interest rate, $R^{fs}_{t+s}(\epsilon^t)$, equals the inverse of $p^{fs}_t(\epsilon^t)$.

Because $\{m_{t+1}(\epsilon^{t+1})\}_{\epsilon^{t+1}}$ affects both short- and longer-term interest rates these rates satisfy cross-restrictions. Consider $R^{f1}_{t+1}(\epsilon^t)$ and $R^{f2}_{t+2}(\epsilon^t)$. By the law of iterated expectations,

$$\begin{aligned}
\left(R^{f2}_{t+2}(\epsilon^t)\right)^{-1} &= \mathbb{E}_t\left[m_{t+1}(\epsilon^{t+1})m_{t+2}(\epsilon^{t+2})\right] = \mathbb{E}_t\left[m_{t+1}(\epsilon^{t+1})\mathbb{E}_{t+1}\left[m_{t+2}(\epsilon^{t+2})\right]\right] \\
&= \mathbb{E}_t\left[m_{t+1}(\epsilon^{t+1})\left(R^{f1}_{t+2}(\epsilon^{t+1})\right)^{-1}\right] \\
&= \left(R^{f1}_{t+1}(\epsilon^t)\right)^{-1}\mathbb{E}_t\left[\left(R^{f1}_{t+2}(\epsilon^{t+1})\right)^{-1}\right] + \mathbb{C}\mathrm{ov}_t\left[m_{t+1}(\epsilon^{t+1}), \left(R^{f1}_{t+2}(\epsilon^{t+1})\right)^{-1}\right].
\end{aligned}$$

Accordingly, there are two drivers of (the inverse of) the long-term interest rate, $R^{f2}_{t+2}(\epsilon^t)$. First, current and expected future (inverse) short-term rates. Second, a *term premium* which reflects the covariance between short-term interest rates and consumption in the subsequent period. When this covariance is positive then the right-most term in the last line is negative and the long-term interest rate exceeds the product of the (expected) short-term rates (and vice versa). Intuitively, a higher than expected short-term interest rate in the future is associated with a capital loss on the long-term bond. When such capital losses tend to occur in periods with high consumption and gains in periods with low consumption then the long-term bond is a good hedge and this lowers its average equilibrium return. The *expectations hypothesis* proposes that the term premium is negligible.

To compare the returns on bonds of different maturity it is useful to express them in normalized form, over time intervals of the same length (e.g., on an annual basis). The *term structure* of interest rates,

$$\left\{R^{f1}_{t+1}(\epsilon^t), \sqrt[2]{R^{f2}_{t+2}(\epsilon^t)}, \sqrt[3]{R^{f3}_{t+3}(\epsilon^t)}, \ldots\right\},$$

collects the normalized returns, and the *yield curve* illustrates the term structure by plotting the normalized returns against maturity. The yield curve is upward sloping when the normalized return increases in maturity, for example because investors expect short-term interest rates to rise in the future in response to changing fundamentals.

5.5 Asset Prices in an Endowment Economy

Every model with a consumption-savings margin (or more generally, an intertemporal margin) is a model of asset prices; equilibrium consumption implies an asset pricing kernel which allows to price arbitrary assets. In an endowment economy with homogeneous households the pricing kernel is particularly straightforward to derive as we now show.

5.5.1 Economy

Consider an economy with a continuum of mass one of infinitely lived homogeneous households who own a fixed capital stock that consists of a continuum of mass one of *trees*. Dividends—the *fruit* of the trees—are exogenous, stochastic, and cannot be stored. They are the only source of income for the households. The budget constraint of household h reads

$$c_t^h(\epsilon^t) + p_t^{tr}(\epsilon^t)\left(tr_{t+1}^h(\epsilon^t) - tr_t^h(\epsilon^{t-1})\right) = tr_t^h(\epsilon^{t-1})d_t^{tr}(\epsilon^t),$$

where $c_t^h(\epsilon^t)$ denotes consumption; $p_t^{tr}(\epsilon^t)$ the tree price; $tr_{t+1}^h(\epsilon^t)$ the household's stock of trees between t and $t+1$; and $d_t^{tr}(\epsilon^t)$ the dividend.

5.5.2 General Equilibrium

While an individual household perceives its asset holdings and consumption to be endogenous, market clearing requires that each household owns one tree, $tr_{t+1}^h(\epsilon^t) = 1$, and consumes the dividend in full. The market price of trees supports this choice. Absent bubbles, it satisfies

$$p_t^{tr}(\epsilon^t) = \mathbb{E}_t\left[\sum_{s=1}^{\infty}\beta^s \frac{u'(d_{t+s}^{tr}(\epsilon^{t+s}))}{u'(d_t^{tr}(\epsilon^t))} d_{t+s}^{tr}(\epsilon^{t+s})\right].$$

Using the equilibrium pricing kernel we may also determine the price of arbitrary other assets, including those that are in zero net supply and not actually traded. For example, the price of a risk-free one period bond that pays off unity is given by

$$p_t^{f1}(\epsilon^t) = \mathbb{E}_t\left[\beta\frac{u'(d_{t+1}^{tr}(\epsilon^{t+1}))}{u'(d_t^{tr}(\epsilon^t))}\right],$$

and the price of an *option* that gives the right to sell a tree in the subsequent period at price $\bar{p}$ equals

$$p_t^{option}(\epsilon^t) = \mathbb{E}_t\left[\beta\frac{u'(d_{t+1}^{tr}(\epsilon^{t+1}))}{u'(d_t^{tr}(\epsilon^t))}\max\left[0, \bar{p} - p_{t+1}^{tr}(\epsilon^{t+1})\right]\right].$$

The capital stock in an economy may be viewed as a fruit yielding tree and we may thus associate the tree price with a broad measure of stock prices. According to that interpretation the excess return on the tree equals the expected *equity premium*.

5.6 Bibliographic Notes

The CAPM is due to Sharpe (1964), Lintner (1965), and Mossin (1966); see also Ross (1976). The C-CAPM is due to Lucas (1978) and Breeden (1979).

Tirole (1982) rules out speculative price bubbles in economies with a finite number of investors with common knowledge of rational expectations and common priors (but possibly different information sets). Tirole (1985) analyzes bubbles in overlapping generations economies with or without assets that generate rents.

Campbell (1986) characterizes stock prices and the term structure of interest rates when consumption is autoregressive log-normal. Cox, Ingersoll and Ross (1985) analyze a general equilibrium model of the term structure.

The model of asset prices in an endowment economy is due to Lucas (1978).

Related Topics Bewley (1980) studies bubbly money in a stochastic incomplete markets environment (see sections 4.4 and 9.4).

Santos and Woodford (1997) analyze rational asset price bubbles in environments with symmetric information, incomplete markets, and incomplete participation.

Cochrane (2001), Cvitanić and Zapatero (2004), Duffie (2001), and LeRoy and Werner (2014) cover financial economics and asset pricing. Magill and Quinzii (1996) cover equilibrium in economies with incomplete financial markets and heterogeneous agents; see also section 8.3. Brunnermeier and Oehmke (2013) and Martin and Ventura (2018) cover asset price bubbles.

See appendix B.4 for asset pricing implications of non-expected utility.

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