Problem Set 1

The Neoclassical Growth Model with Population Growth and Government Spending

Exercise 1: The (growing) Household Problem

Consider a representative family. It maximizes $\sum_{t=0} \beta^t N_t u(c_t)$ subject to the dynamic budget constraint $a_{t+1}(1+n_{t+1}) = a_t R_t + w_t - c_t - \tau_t w_t$, subject to a no-Ponzi-game condition $\lim_{t\to\infty} q_t a_{t+1} \nu_{t+1} \geq 0$ where a_t are asset holdings (negative values are debt), R_t is the (real) gross interest rate, τ_t are proportional labour taxes, c_t is consumption, the net population rate is given by $(1+n_{t+1}) \equiv N_{t+1}/N_t$ and $q_t \equiv (R_1 R_2 \dots R_t)^{-1}$. Observe that the individual labour supply is exogenously set to one.

(a) Write down the maximization problem and derive the Euler equation.

(b) Show that the IBC of the household is given by

$$0 \leq a_0 R_0 + \sum_{t=0}^{\infty} (w_t(1-\tau_t) - c_t)\nu_t q_t,$$

where $\nu_t \equiv (1+n_1)(1+n_2)\cdot \ldots \cdot (1+n_t), \nu_0 = 1, t \geq 0$.

c) Assume $R_t = \beta^{-1}, n_t = n$, $\tau_t = \tau$ and $w_t = w$ $\forall t$. Compute consumption c_t and the evolution of savings a_t.

(d) Explain why the household problem is only well-defined if $\rho > n$ where $\beta \equiv \frac{1}{1+\rho}$?

Exercise 2: Equilibrium and Dynamics

Consider an economy where the only asset that households can save in is capital. Households can rent their capital and labour to a representative firm at prices r_t and w_t respectively. The firm produce according to a Cobb-Douglas production function $f(K_t, L_t) = A_t K_t^{\alpha} L_t^{1-\alpha}$ and maximizes its profits. Profits are redistributed to households as dividends and capital depreciates at the rate δ which implies $R_t = 1 + r_t - \delta$. There exists a government that taxes labour and spends g_t such that the government budget constraint is given by $g_t = \tau_t w_t$. Finally, assume that population growth is constant.

(a) Find the equilibrium conditions.

(b) Derive and plot the steady state resource constraint and Euler equation (with k on the horizontal and c on the vertical axis).

(c) Starting from the steady state, analyse the response of the economy to a temporary (g increase in t_1 and decreases again in t_2) or permanent increase (g increases in t_1 forever) in government spending by means of a phase diagram.

(d) Repeat (a)-(c) with capital income taxes, i.e. when $g = k_t R_t \tau_t$.

(e) In what sense is the labour tax more efficient than the capital tax in this economy? Explain why this is the case, keeping in mind that households typically do not take the government budget into account when making their consumption and investment choices.

Exercise 3: Ricardian Equivalence

Assume there is no population growth. Starting from the steady state in $t = 0$, the following "shock" hits the economy. Agents learn that in $t = 0$, tax revenue is $\tilde{\tau}_0 w_0 = \tau w - \Delta$, in $t = T$ tax revenue is $\tilde{\tau}_T w_T = \tau w + R^T \Delta$ and $\tilde{\tau}_t \tilde{w}_t = \tau w$ for all $t \neq 0, T$. Assume $g = \tau w$. The government finances the budget deficit with debt b_t (which in equilibrium needs to pay R_t). Show that the allocation (capital and consumption) remains unchanged. Why?

Problem Set 2

Government debt and social security

Exercise 1: Debt in the OLG Model

Consider an OLG model where households live for two periods, the population at time is N_t and growths at rate $v_t = \frac{N_t}{N_{t-1}}$, the households supply exogenously one unit of labour when one, pay labour income taxes τ_t and can save for retirement by owning capital, k_{t+1} or government debt, b_{t+1}. In retirement, households no longer work but can consume out of their savings. The household's problem is

$$\max_{c_{1,t},c_{2,t+1},b_{t+1},k_{t+1}} u(c_{1,t}) + \beta u(c_{2,t+1})$$

$$\text{subject to} \quad \nu(b_{t+1} + k_{t+1}) = w_t(1 - \tau_t) - c_{1,t}$$
$$c_{2,t+1} = \nu b_{t+1} R^b_{t+1} + \nu k_{t+1} R^k_{t+1} + \nu \pi_{t+1},$$

where $c_{1,t}$ is consumption of the young, $c_{2,t}$ consumption of the old (retired) household, R^k_{t+1} is the return paid on capital, R^b_{t+1} is the return paid on government bonds and π_{t+1} is profits by firms. Assume that firms have access to a Cobb-Douglas production. Capital is build by young agents and depreciated fully after they die. The government's budget constrained is given by

$$g_t + b_t R^b_t = \nu b_{t+1} + w_t \tau_t.$$

(a) Derive the Euler Equations and show that in equilibrium $R^b_{t+1} = R^k_{t+1}$ given that the government has positive debt at all times.

(b) The firm solve the following profit maximization problem:

$$\max_{N_t,k_t} (N_t k_t)^\alpha N_t^{1-\alpha} - k_t N_t r_t - w_t N_t$$

Derive the firm's first order conditions.

(c) Derive the aggregate resource constraint.

(d) Show that for $u(c) = \ln(c)$ the savings of a young household in period t are given by $s w_t(1 - \tau_t)$ where $s = \beta/(1 + \beta)$.

Now consider that instead of issuing debt, the government runs a pay-as-you-go fully funded social security system, maintaining the other assumptions of logarithmic preferences and full depreciation. Old agents receive transfers T_t which are fully funded by labour taxes, such that the governments budget constraint is given by $T_t = w_t \tau_t^s$. (i.e. Transfers per old agent is νT_t).

(e) Derive the Euler Equation.

(f) Show that savings of a young household are given by $s_t w_t (1-\tau_t^s)$ where

$$s_t = \frac{\beta}{1+\beta} - \frac{\nu T_{t+1}}{R_{t+1}(1+\beta)w_t(1-\tau_t^s)}.$$

(f) For any social security policy $\{\tau_t^s, T_{t+1}\}_{t\geq 0}$ there exists an associated debt policy $\{\tau_t, b_{t+1}\}_{t\geq 0}$ that supports the same allocation (among the two systems considered in this exercise). Assume $g_t = 0$ in both systems. Find the conditions needed for equivalence.

Exercise 2: Debt as a bubble

Consider a balanced growth path (or the steady state) with constant capital-labor ratio k and debt-labor ratio b

(a) State the government budget constraint and solve for government expenditures.

(b) Show that if the economy is dynamically inefficient, the government can finance expenditures ($g > 0$) without collecting taxes ($\tau = 0$).

(c) Show that in this case the No-Ponzi-Game condition is violated.

Problem Set 3

Taxation, seigniorage and cash-in-advance

Exercise 1: Multiple Tax Instruments

A representative household has preferences $\sum_{t=0}^{\infty}\beta^t u(c_t, x_t)$ where c_t is consumption and x_t is leisure time. The households have one unit of time per period to either use for work or leisure. The government taxes labor income at rate τ_t^w, consumption expenditure at rate τ_t^c and savings income at rate τ_t^k. The intertemporal budget constraint is given by

$$a_0 R_0(1-\tau_0^k) + \sum_{t=0}^{\infty} q_t \kappa_t \left(w_t(1-x_t)(1-\tau_t^w) - c_t(1+\tau_t^c)\right) = 0,$$

or equivalently

$$\frac{a_0 R_0(1-\tau_0^k)}{1+\tau_0^c} + \sum_{t=0}^{\infty} q_t \kappa_t \xi_t \left(w_t(1-x_t)\frac{1-\tau_t^w}{1+\tau_t^c} - c_t\right) = 0,$$

where $\kappa_t \equiv [(1-\tau_1^k)\cdots(1-\tau_t^k)]^{-1}$ and $\xi_t \equiv (1+\tau_t^c)/(1+\tau_0^c)$.

(a) Write down the problem of the household and derive the household's two optimality conditions.

(b) There are two wedges, one distorting the intratemporal margin and the other the intertemporal one. Identify them. Show that different policy regimes, say $\mu = \{\tau_t^k, \tau_t^c, \tau_t^w\}_{t=0}^{\infty}$ and $\tilde{\mu} = \{\tilde{\tau}_t^k, \tilde{\tau}_t^c, \tilde{\tau}_t^w\}_{t=0}^{\infty}$, leave the household's choice set unchanged as long as the two wedges are the same.

(c) Assume that initially the government uses all three tax instruments at its disposal, i.e. it uses policy $\mu = \{\tau_t^k, \tau_t^c, \tau_t^w\}$, where $\tau_0^k = 0$, to finance a stream of government expenditures $\{g_t\}$. Find the restrictions on a policy $\tilde{\mu} = \{\tilde{\tau}_t^w, \tilde{\tau}_t^c\}$, where $\tilde{\tau}_0^c = \tau_0^c$, that can implement the same equilibrium and does not use savings income taxes.[1]

Exercise 2: Financing by Seignorage

Consider an economy where the government does not issue debt and a constant deficit, $g - \tau > 0$ is financed by seignorage only. The governments dynamic budget constraint then simplifies to

[1] You only need to check that the household budget set remains the same as in b) as this implies that the government budget constraint is also satisfied for the new policy.

$$g - \tau = \frac{M_{t+1} - M_t}{P_t} = \frac{M_{t+1} - M_t}{M_t} \frac{M_t}{P_{t-1}} \frac{P_{t-1}}{P_t}.$$

Assume that the government chooses a constant money growth rate γ, i.e. $M_{t+1} = \gamma M_t$. Further assume that real activity in the economy is constant and money demand depends negatively on expected inflation. In particular the demand for money takes the form

$$\frac{M_{t+1}}{P_t} = \alpha^2 - \beta^2 \mathbb{E}_t \Pi_{t+1},$$

where $\alpha, \beta > 0$. On a balanced growth path, the inflation and expected inflation coincide and the inflation rate is the same as the money growth rate.

a) What is the money growth rate that maximizes seignorage revenue? Draw a Laffer curve.

b) How large can the deficit be such that there is a low and a high inflation equilibrium?

Exercise 3: Economy with cash-in-advance constraint

Consider a deterministic economy with an infinitely lived representative household and a government. Preferences are given by $\sum_{t=0}^{\infty} \beta^t u(c_t)$. The structure of the model is as follows: at the beginning of each day, agents wake up with some real asset, b_t which yield a real interest rate, $R_t > 1$, some nominal government bonds, B_t, which yield a return nominal return, $I_t \geq 1$ and money, M_t, which does not yield any return. They can then reallocate their portfolio. In the afternoon, two things happen: First, the households go shopping for consumption goods for which they require money which they have accumulated in the morning. Thus, we impose a cash-in-advance constraint: $c_t \leq M_{t+1}^h / P_t$. Second, the households receive some endowment, w_t, which they sell. Thus, the household's dynamic budget constraint reads

$$\tau_t + b_{t+1} + \frac{B_{t+1}}{P_t} + \frac{M_{t+1}^h}{P_t} = b_t R_t + \frac{B_t I_t}{P_t} + \frac{M_t^h - P_{t-1}(c_{t-1} - w_{t-1})}{P_t}$$

The government's budget constraint can be written in a similiar fashion assuming that the government spends g_t in the afternoon and taxes lump sum, τ_t, in the morning:

$$b_t R_t + \frac{B_t I_t}{P_t} + \frac{M_{t+1}^g}{P_t} = \tau_t + b_{t+1} + \frac{B_{t+1}}{P_t} + \frac{M_t^g - P_{t-1} g_{t-1}}{P_t} + \frac{M_{t+1} - M_t}{P_t}$$

Furthermore, by definition $M_t \equiv M_t^h + M_t^g$. Lastly, we impose a cash-in-advance for the government as well:

$$g_t \leq M_{t+1}^g / P_t$$

(a) Write down the problem of the household and the associated first-order conditions.

(b) Derive the Fisher equation.

(c) Show that the cash-in-advance constraint does not bind if there is sufficient deflation (to be precise if $\frac{P_{t+1}}{P_t} = 1/R_{t+1} < 1$). Why?

(d) Derive the Euler equation.

(e) Derive the aggregate resource constraint.

Problem Set 4

Fiscal Theory of the Price Level and Financing of Government Spending

Exercise 1: Game of Chicken

Consider a deterministic version of the cash-in-advance model discussed in the previous problem set. The aggregate resource constraint is given by $\omega_t = c_t + g_t$. The Euler equation is given by: $u'(c_t) = \beta R_{t+1} u'(c_{t+1})$. The cash-in-advance constraint is binding: $M_{t+1}/P_t = \omega_t$. Finally, assume that endowments $\omega_t = \omega$ and government expenditures $g_t = g$ are constant.

a) Combine the resource constraint of the economy and the Euler equation of the household to show that $1/R_t = \beta \ \forall t$.

Assume the fiscal authority issues real debt and levies taxes τ_t, and a monetary authority chooses money supply. At the beginning of period 0, there is outstanding debt $b_0 R_0$ and outstanding money balances M_0. The government budget is given by:

$$b_t R_t + g = \tau_t + \frac{M_{t+1} - M_t}{P_t} + b_{t+1}.$$

b) Show that the IBC of the government (consisting of the fiscal and the monetary authority) is

$$b_0 R_0 = \sum_{j=0}^{\infty} \beta^j (\tau_j - g) + \omega \sum_{j=0}^{\infty} \beta^j (1 - \Pi_j^{-1}),$$

where $\Pi_j = P_j / P_{j-1}$ and assuming that $\lim_{t\to\infty} b_{t+1}\beta^t = 0$.

c) Assume the fiscal authority moves first and commits to a sequence of taxes $\{\tau_t\}_{t=0}^{\infty}$. Find the stabilizing inflation rate (where Π_j is constant) that fulfills the IBC. Show that Π increases in response to lower taxes.

d) Assume the monetary authority moves first and chooses the money supply such that the inflation rate is zero. What are the restrictions placed on the fiscal policy if we assume that $\tau_t = \tau$ for all t?

Exercise 2: Fiscal Theory of the Price Level

Consider a two period ($t = 0, 1$) model with nominal debt and no money balances or seignorage. The household's dynamic budget constraint is

$$\frac{B_{t+1}}{P_t} + c_t = \omega_t - \tau_t + \frac{B_t I_t}{P_t},$$

where ω_t is an endowment, τ_t taxes, B_t nominal bonds and I_t the nominal interest rate. The resource constraint is given by $\omega_t = c_t + g_t$, with g_t given exogenously. Initially there is no outstanding debt, $B_0 = 0$ and $B_2 = 0$.

a) Derive the government's dynamic budget constraint.

b) Explain the difference between a Ricardian and a non-Ricardian regime. Which regime does the Fiscal Theory of the Price Level assume?

c) The proponents argument: Show that under the Fiscal Theory of the Price Level, the government can choose an arbitrary τ_1 in the *last* period. What is P_1?

d) The opponents argument: Assume that the following two Euler equations for both government bonds and some real bonds hold:

$$u'(c_0) = u'(c_1)\beta R_1,$$

$$u'(c_0) = u'(c_1)\beta \frac{I_1}{\Pi_1}$$

Derive the following pricing equation for government bonds:

$$\frac{B_1}{P_0} = \frac{1}{R_1}\frac{B_1 I_1}{P_1}.$$

Use the government budget constraint in period $t = 0$ and the equations above and show that the intertemporal budget constraint for the government has to hold for any price level and that τ cannot be set arbitrarily. Discuss the two arguments.

Exercise 3: Tax Collection Costs in a Small Open Economy

Consider a small open, two-period lived economy with a fixed interest rate R. Households and the government can lend and borrow freely at this interest rate. The government needs to finance spending g_t per period and it levies a labour tax τ_t on the household's exogenous income w_t. The government's budget constraint is

$$g_t + b_t R_t = \tau_t w_t + b_{t+1},$$

where $b_0 = 0$.

There are administrative costs to raising taxes. These costs are an increasing, nonlinear function $\Phi(T_t)$ of the level of total tax revenues T_t, such that

$$\Phi(T_t) = \varphi_1 T_t + \frac{\varphi_2}{2} T_t^2, \qquad \varphi_1, \varphi_2 > 0.$$

The tax collection costs are payed for by the household, such that her after-tax income is $w_t(1-\tau_t)-\Phi(T_t)$. Accordingly, the household's dynamic budget constraint is

$$c_t + a_{t+1} = w_t(1-\tau_t) - \Phi(T_t) + R_t a_t.$$

Further, let $a_0 = 0$.

a) Write down the household's and the government's intertemporal budget constraints.

b) Interpret the two restrictions imposed on φ_1 and φ_2.

c) Characterize the optimal behaviour of the government. What is the level of total tax raised in each period? Interpret.

   Hint: Since the household has access to financial markets, her net wealth computed at the exogenous interest rate is a sufficient statistic for her lifetime utility. Accordingly the optimal policy maximizes the household's net wealth subject to the government's intertemporal budget constraint.

d) Show that, in general, the government will run a budget surplus in one period and a budget deficit in the other period. What is the effect of an increase in R on taxes T?

Problem set 5

Ramsey Taxation

Exercise 1: The Primal Approach

Consider the following two period model, $t = 0, 1$ where households have preferences over consumption c_t and leisure x_t:

$$U = \sum_{t=0}^{1} \beta^t \left(c_t - \frac{1}{2}(1 - x_t)^2 \right).$$

The household is endowed with one unit of time and divides it between labor n and leisure x such that $n + x = 1$. The production function is linear in labor only, i.e. $y_t = n_t$ and it follows that $w_t = 1$.[1] The government has to finance an exogenous government spending stream $\{g_0, g_1\}$ and levies a tax τ_t on the household's labor income and can issue debt, b_1, at the interest rate R.

(a) Derive the household's intertemporal budget constraint.

(b) Derive the government's intertemporal budget constraint.

(c) Derive the resource constraint.

(d) Find the household's optimality conditions (Euler Equation and Labour/Leisure Choice).

From now on, we assume that we have an interior solution and therefore that $\beta R = 1$.

(e) Proof that the Social Planner Allocation is not implementable. [2]

(f) Find the implementability constraint. [3]

(g) Show that the government optimally smoothes tax rates over time, $\tau_0 = \tau_1$ using the primal approach.[4]

(h) Assume $g_0 = g_1 = 1/4$. Solve for the equilibrium allocation.

[1]This follows directly from the fact that under perfect competition the marginal product of labour, which is one in this case, equals the wage.

[2]The Social Planner Allocation can be found by maximizing the agent's utility subject to the resource constraints.

[3]The implementability constraint is found by combining the agent's budget constraint and their optimality conditions.

[4]Maximize the agent's utility subject to the ressource constraint and the implementability constraint.

Exercise 2: The Dual Approach

Consider the same setup as in Exercise 1. Now we follow the dual approach where the Ramsey program is a choice of prices (here taxes) rather than allocation.

(a) Using your the optimality condition and the resource constraints derived in the previous exercise, write the household's indirect utility function[5] and the government's intertemporal budget constraint as a function only of tax rates and government expenditure (that is, eliminate x).

(b) Show that the government optimally smoothes tax rates over time, $\tau_0 = \tau_1$ using the dual approach.

Exercise 3: Risk and Complete Markets

Consider the same setup as above with a few changes. Preferences are now given by

$$U = \mathbb{E}_0 \sum_{t=0}^{1} \beta^t \left(\log(c_t) - \frac{1}{2}(1 - x_t)^2 \right)$$

and there are now two possible states in period 1. In particular, there is a high and a low state with probabilities π^h, π^l, and government consumption g_1^h, g_1^l and taxes τ_1^h, τ_1^l can differ across states (there is no uncertainty in period 0). The government issues state-contingent debt b_1^h and b_1^l. The variable b_1^s denotes a claim on the government and entitles the household to one unit of the consumption good in period 1 if state $s = h, l$ has occurred. Let the price of this debt in t=0 be q_0^s for $s = h, l$. The household's budget constraints in period 0 and 1 respectively are

$$c_0 + q_0^h b_1^h + q_0^l b_1^l = (1 - \tau_0)(1 - x_0) \quad \text{and}$$
$$c_1^s = (1 - \tau_1^s)(1 - x_1^s) + b_1^s \quad s = h, l.$$

Everything else is unchanged and as in Exercise 1.

(a) Combine the household's DBCs and derive the households intertemporal budget constraint.

(b) Derive the household's optimality conditions.

(c) With your result in (b), derive the implementability constraint.

(d) Derive the first-order conditions of the government's program. Proof that $\tau_0 = \tau_1^h = \tau_1^l$.

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[5]The indirect utility represents the households utility as a function of price. In this case, the tax rates.

Problem Set 6

Time Consistent Monetary Policy

Exercise 1: Barro and Gordon (1983)

Consider the following reduced-form loss function for a central bank:

$$Z_0 = \sum_{t=0}^{\infty} \beta^t z_t \quad \text{where} \quad z_t = \frac{a}{2}\pi_t^2 - b(\pi_t - \pi_t^e), \; a, b, > 0.$$

Here, z_t, π_t and π_t^e denote losses to the central bank, inflation, and inflation expectations formed by the private sector at the beginning of a period t, respectively. Deviations of inflation from some baseline value (here zero) generate losses for the central bank since they give rise to relative price distortions or because the public dislikes inflation. Realizations of inflation in excess of inflation expectations generate gains because they stimulate output (due to a Phillips curve relationship in the background) or devalue real public debt (and thus, help reduce distorting taxes). Inflation is under the direct control of the central bank.

(a) With discretion, the central bank chooses inflation after the private sector has formed inflation expectation in every period. What is the inflation rate π_t^d that minimizes the central bank's loss function? What is the loss z_t^d (as a function of a and b).

(b) Assume the central bank can commit to implementing an inflation rate. Effectively, the central bank then chooses both the inflation rate and inflation expectations. What is the optimal inflation rate in equilibrium? What is the loss z_t in every period? Is commitment superior to discretion?

(c) Assume the rule of the central bank is to follow $\pi_t = \pi_t^e = 0$. Show that the central bank is tempted to deviate. What is the loss z_t if the central bank does deviate?

The central bank can be induced to follow a rule $\pi_t = \pi^*$ because of reputation concerns. Suppose that the private sector expects the central bank to implement inflation according to the announced rule if expectations were correct in the previous period; if expectations were wrong then the public expects the central bank to implement the discretionary policy. That is

$$\pi_t^e = \pi^* \text{ if } \pi_{t-1} = \pi_{t-1}^e$$
$$\pi_t^e = \pi_t^d \text{ if } \pi_{t-1} \neq \pi_{t-1}^e.$$

A deviation of inflation from the rule thus triggers a one period "punishment phase" where the public expects the discretionary outcome and the central bank responds accordingly. After the punishment period the public expects the central bank to revert to the rule. A rule can then be sustained if the temptation to renege, given the inflation announcement π^* is smaller than following the rule. In other words, the gain from deviating in period t is smaller than the discounted loss from having to implement π^d_{t+1} in period $t+1$ in the "punishment phase".

(d) Show that a rule π^* can be sustained if it fulfills the restriction

$$\beta \frac{a}{2}\left(\left(\frac{b}{a}\right)^2 - (\pi^\star)^2\right) \geq \frac{a}{2}\left(\frac{b}{a} - \pi^\star\right)^2 .$$

(e) Show that the best enforceable rule, that is the smallest $\pi^\star$ that satisfies the restriction is

$$\pi^* = \frac{b}{a}\frac{1-\beta}{1+\beta}$$

What happens if $\beta \to 0$?