

Macroeconomic Models

Dirk Niepelt

Study Center Gerzensee; University of Bern

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Preliminary, incomplete, to be improved (please bring errors to the author's attention)

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Introduction

This book introduces the workhorse models of modern macroeconomics. It focuses on concepts and frameworks. The target audience are master students or beginning doctoral students in economics who are familiar with microeconomics and calculus at the undergraduate level.

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Chapter 1

Microeconomic Foundations

Modern macroeconomics uses the concepts and tools of microeconomic general equilibrium theory. It describes economic outcomes as the result of optimizing choices by households and firms that interact on markets, over time, subject to affordability and feasibility constraints.

This micro founded approach to macroeconomics contrasts with macroeconomic frameworks often taught at the undergraduate level, for example the IS-LM model. The latter relies on ad-hoc assumptions about the relationship between consumption and income or investment and the interest rate, say. It does not build on assumptions about microeconomic primitives such as preferences and technology.

To prepare for the subsequent micro founded macroeconomic analysis, we review important microeconomic concepts and introduce assumptions about the primitives. We also provide a perspective on the history of macroeconomic thought since the late 1950s.

1.1 Microeconomics

1.1.1 Allocation, Feasibility, Optimality

An *allocation* consists of a consumption vector for each household and a net production vector for each firm. For example, an allocation in an economy with one household, one firm and three goods could be $\{(1, 2, 1), (-1, 1, -1)\}$: The household consumes one unit each of the first and third good and two units of the second good, while the firm uses one unit each of the first and third good as inputs and supplies one unit of the second good. In a model with government the allocation also includes a consumption and production vector for the government. In an open economy model the allocation also includes a consumption and production vector for the “rest of the world.”

An allocation is *feasible* if for each good, total consumption does not exceed the endowment plus net production. For example, the allocation given above is feasible if the endowment vector equals $(2, 1, 2)$ but it is not feasible if the endowment vector equals $(2, 2, 1)$.

A feasible allocation Pareto dominates another feasible allocation if at least one

household strictly prefers the former and no household strictly prefers the latter. A feasible allocation is *Pareto optimal* or Pareto efficient if it is not Pareto dominated by any other feasible allocation. The set of Pareto optimal allocations traces the Pareto frontier.

1.1.2 Competitive Equilibrium

The consumption set of a household contains all consumption vectors that the household conceivably could consume in the absence of budgetary restrictions. For example, the consumption set might exclude negative quantities of apples.

The budget set of a household contains all consumption vectors in the household's consumption set that the household can afford to consume. The budget set is determined by household endowments, the prices of all goods, and firm profits which are distributed to households according to prespecified ownership rights.

The production set of a firm contains all production vectors that are feasible given the firm's technology.

A *competitive* or *Walrasian equilibrium* is an allocation and a set of prices satisfying three conditions: First, the allocation is feasible. Second, conditional on production sets and taking prices as given, each firm's production choice is profit maximizing. And third, conditional on preferences and taking prices and firm profits as given, each household's consumption choice is utility maximizing in the household's budget set.

The equilibrium is "competitive" because firms and households take prices as given. Alternative, non-competitive equilibria might exist as well where agents perceive their choices to affect prices or firm profits, and they exploit this feature. For example, a firm might want to reduce output in order to raise the equilibrium price of its product. We mostly abstract from non-competitive behavior and focus on competitive equilibria.

A competitive equilibrium with lump-sum transfers between households that sum to zero is referred to as price equilibrium with transfers.

1.1.3 Walras' Law

Let p denote the vector of prices across goods. Let $z_g^h(p)$ denote household h 's net demand function for good g that is, desired consumption net of the household's endowment and share of firm profits, as a function of p . Let $z^h(p)$ denote the vector of household h 's net demand functions across goods. Let $z_g(p) \equiv \sum_h z_g^h(p)$ denote the aggregate excess demand function for good g . Finally, let $z(p) \equiv \sum_h z^h(p)$ denote the vector of excess demand functions across goods. If all households satisfy their budget constraints, then $p \cdot z^h(p) = 0$ for all h . By implication, *Walras' Law* holds: The values of excess demands sum to zero, $p \cdot z(p) = 0$.

Walras' Law implies that with strictly positive prices, feasibility of an equilibrium allocation is equivalent to market clearing in all markets. To see this, note that optimization and feasibility imply $z(p) \leq 0$. If a good has strictly positive price, excess demand for that good therefore must be zero (otherwise, $p \cdot z(p) \neq 0$). Walras' Law also implies that with strictly positive prices, market clearing in all markets but one

requires market clearing in the remaining market. To see this, suppose all markets except market j clear, $z_g(p) = 0$ for all $g \neq j$, such that $\sum_{g \neq j} p_g z_g(p) = 0$. Since $p_j > 0$, $z_j(p)$ also must equal zero (otherwise, $p \cdot z(p) \neq 0$).

1.1.4 Fundamental Theorems of Welfare Economics

The fundamental theorems of welfare economics relate equilibrium allocations and Pareto optimal allocations. The *first fundamental theorem of welfare economics* formalizes the notion of an “invisible hand.” It states that, if an allocation and price system constitute a price equilibrium with transfers (in particular, a competitive equilibrium), then the allocation is Pareto optimal as long as certain assumptions (spelled out below) are satisfied. Decentralized choices by price taking individuals thus are consistent with Pareto optimality, and this holds true even if lump-sum transfers occur before market transactions take place.

Let $\{\{x^h\}_h, \{y^f\}_f\}$ be an allocation conditional on $\{e^h\}$ where x^h , e^h , and y^f denote household h 's consumption vector, the household's endowment vector, and firm f 's net production vector respectively. The allocation is feasible if $\sum_h (x^h - e^h) \leq \sum_f y^f$. Let $\{\{x^{h*}\}_h, \{y^{f*}\}_f, p^*\}$ be a competitive equilibrium where p^* denotes the equilibrium price vector. The theorem claims that no feasible allocation Pareto dominates $\{\{x^{h*}\}_h, \{y^{f*}\}_f\}$. The proof by contradiction proceeds in three steps. (i) Suppose that a feasible Pareto dominating allocation, $\{\{x^{h\bullet}\}_h, \{y^{f\bullet}\}_f\}$ say, does exist. Some household then strictly prefers the consumption vector in the Pareto dominating allocation over the consumption vector in the equilibrium allocation. But since the household chooses optimally, the former must lie outside the household's budget set implied by p^* . (ii) Summing over all households implies

$$\sum_h p^* \cdot (x^{h\bullet} - e^h) > \sum_f p^* \cdot y^{f*}.$$

(iii) Since firms maximize profits, $\sum_f p^* \cdot y^{f*} \geq \sum_f p^* \cdot y^{f\bullet}$. Combining the inequalities yields

$$\sum_h p^* \cdot (x^{h\bullet} - e^h) > \sum_f p^* \cdot y^{f\bullet}.$$

But feasibility of $\{\{x^{h\bullet}\}_h, \{y^{f\bullet}\}_f\}$ implies the reverse inequality. We have therefore arrived at a contradiction which proves the theorem.

The first step of the argument requires preferences to be locally non-satiated, households and firms competitive, and markets complete (all goods are traded). The second step requires the market value of endowments to be finite which is guaranteed, for example, if the number of households is finite.

The *second fundamental theorem of welfare economics* formalizes the notion that Pareto optimal allocations can be decentralized through markets and lump-sum transfers. It states that for every Pareto optimal allocation, there exists a set of prices such that the allocation and the prices constitute a price equilibrium with transfers. Necessary as-

assumptions of the theorem include convex production sets as well as convex and locally non-satiated preferences.

1.2 Primitives

The primitives of a modern macroeconomic model are those of a microeconomic model, specifically preferences and technology. Since macroeconomic models often feature dynamic and stochastic environments we review the event tree and discuss how preferences can be specified in such settings. We also review the neoclassical production function.

1.2.1 Event Tree

Time is denoted by t . It runs from zero, the initial date, to some final date T or to infinity. We denote by ϵ^t the *history* of realizations of exogenous shocks up to and including date t . At the initial date, history ϵ^0 is known but history $\epsilon^t, t \geq 1$, is not known if shocks hit the economy between “now” and “then.”

The *event tree* in figure 1.1 illustrates a three-period example. At date $t = 1$, one of two possible shock realizations occurs, “up” (for example promotion) or “down” (demotion). At date $t = 2$, again one of two possible realizations occurs. History ϵ^1 thus takes two values and history ϵ^2 four. An economic choice or outcome at the initial date, y_0 say, is deterministic. In contrast, the choice or outcome at date $t = 1$ is a random variable that takes two values, $y_1(\text{up})$ or $y_1(\text{down})$. Similarly, the random variable $y_2(\epsilon^2)$ takes four values, $y_2(\text{up, up})$, $y_2(\text{up, down})$, $y_2(\text{down, up})$, or $y_2(\text{down, down})$.

In deterministic settings, we index variables by time; history does not add relevant information in this case. In stochastic settings, variables need to be indexed by history to avoid ambiguity as the time index does not carry all relevant information. But when there is no danger of confusion we sometimes suppress history to simplify the notation. Also, when indexing by history there is in principle no need to additionally index by time since the former implies the latter. Nevertheless, we always index variables by time to keep the notation uniform.

1.2.2 Preferences

In the simplest dynamic model, households consume a single (composite) commodity, c . They may also consume leisure, x . Households trade off current and future consumption in different histories. This contrasts with the trade-off between “apples” and “oranges” in a static model. However, as we will see in chapter 2 the optimality conditions in dynamic and static settings are isomorphic.

Preferences map a sequence of history-contingent consumption into utility,

$$U \left(c_0, \{c_1(\epsilon^1)\}_{\epsilon^1}, \dots, \{c_T(\epsilon^T)\}_{\epsilon^T}, x_0, \{x_1(\epsilon^1)\}_{\epsilon^1}, \dots, \{x_T(\epsilon^T)\}_{\epsilon^T} \right).$$

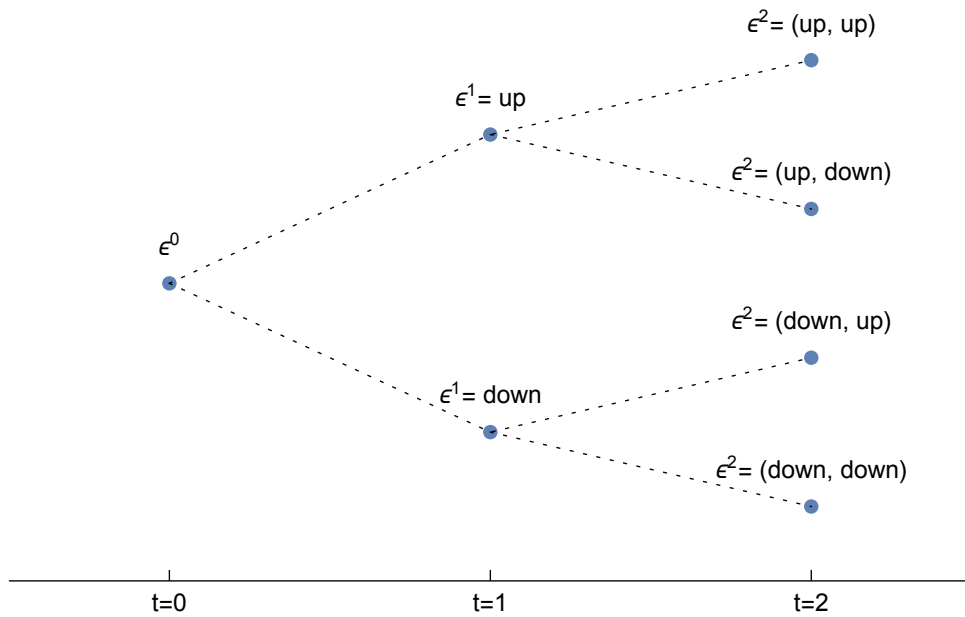


Figure 1.1: Event tree: An economy with three periods and two states of nature each at dates $t = 1, 2$, “up” or “down.” A history ϵ^t at date t describes the sequence of realized states up to and including date t .

The function U increases in all its arguments. The notation $\{c_t(\epsilon^t)\}_{\epsilon^t}$ indicates that goods consumption at date t takes multiple values, depending on the realized history. For example, in the environment illustrated in figure 1.1, we have $\{c_1(\epsilon^1)\}_{\epsilon^1} = \{c_1(\text{up}), c_1(\text{down})\}$, et cetera.

For tractability, we often assume that preferences are additively separable across time and histories that is, U is a weighted sum. The weight attached to date t equals β^t where $\beta \in [0, 1)$ denotes the *psychological discount factor* which measures the degree of patience. The weight attached to a particular history equals the probability that this history occurs. A consumption sequence thus is evaluated according to the discounted *expected utility* that it generates,

$$U = \mathbb{E}_0 \left[\sum_{t=0}^T \beta^t u(c_t(\epsilon^t), x_t(\epsilon^t)) \right].$$

Here, \mathbb{E}_s denotes the mathematical expectation conditional on information available at date s , history ϵ^s . Separability across time and histories implies that the marginal utility of c or x at a date or history is independent of the values of c and x at other dates or histories.

The period utility function u exhibits positive, decreasing marginal utility. Unless otherwise noted, we assume that it is continuously differentiable, marginal utility is strictly decreasing, and consumption of the good and leisure at each date are essential, $\lim_{c \downarrow 0} u_c(c, x) = \lim_{x \downarrow 0} u_x(c, x) = \infty$. Here, $u_c(c, x)$ and $u_x(c, x)$ denote the partial

derivatives $\partial u(c, x)/\partial c$ and $\partial u(c, x)/\partial x$ respectively; we use this notation for partial derivatives throughout the book.

For even more tractability, we sometimes restrict u to be of the constant intertemporal elasticity of substitution (CIES) or equivalently, constant relative risk aversion (CRRA) form. Disregarding leisure for now, u then is given by

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma} \text{ for } \sigma > 0, \sigma \neq 1.$$

This functional form is not defined for $\sigma = 1$ but applying L'Hôpital's rule implies $\lim_{\sigma \rightarrow 1} (c^{1-\sigma} - 1)/(1 - \sigma) = \ln(c)$; the logarithmic function thus constitutes a special case.

As the name suggests, CRRA/CIES preferences exhibit constant relative risk aversion and a constant intertemporal elasticity of substitution. To see the former, recall that the coefficient of relative risk aversion is defined as $-u''(c)c/u'(c)$; with CRRA preferences this reduces to σ . To see the latter, recall that the elasticity of substitution measures how strongly a change of relative price affects relative demand. With CIES preferences, the elasticity of the ratio c_{t+1}/c_t with respect to the relative price of c_{t+1} and c_t reduces to $1/\sigma$. As we will see in chapter 2, CIES preferences simplify the equilibrium conditions in dynamic models.

Figure 1.2 illustrates the role of the parameter σ for CIES preferences. The top row of the figure plots $u(c)$ for $\sigma = 0.01, 0.99, 2.00$ (from left to right); the bottom row plots indifference curves of the utility function $U = u(c_0) + \beta u(c_1)$ for the same set of parameter values (and $\beta = 0.98$).

1.2.3 Technology

Firms employ a production function, f , that maps inputs of physical capital, K , and labor, L , into output. Unless otherwise noted, we assume that the production function is neoclassical. That is, f exhibits strictly positive and diminishing marginal products as well as constant returns to scale (CRTS),

$$f_K(K, L), f_L(K, L) > 0; f_{KK}(K, L), f_{LL}(K, L) < 0; \phi f(K, L) = f(\phi K, \phi L) \forall \phi > 0.$$

We also often assume that f satisfies the Inada conditions

$$\lim_{K \downarrow 0} f_K(K, L) = \lim_{L \downarrow 0} f_L(K, L) = \infty, \quad \lim_{K \rightarrow \infty} f_K(K, L) = \lim_{L \rightarrow \infty} f_L(K, L) = 0.$$

Due to CRTS, output coincides with total factor payments to the suppliers of K and L if the rental rates of K and L equal the respective marginal products. On competitive factor markets this is the case. Also due to CRTS, output per worker as well as marginal products only depend on the capital-labor ratio, $k \equiv K/L$, not on K and L individually. Both these facts follow from Euler's homogeneous function theorem. The Inada conditions help guarantee interior equilibria.

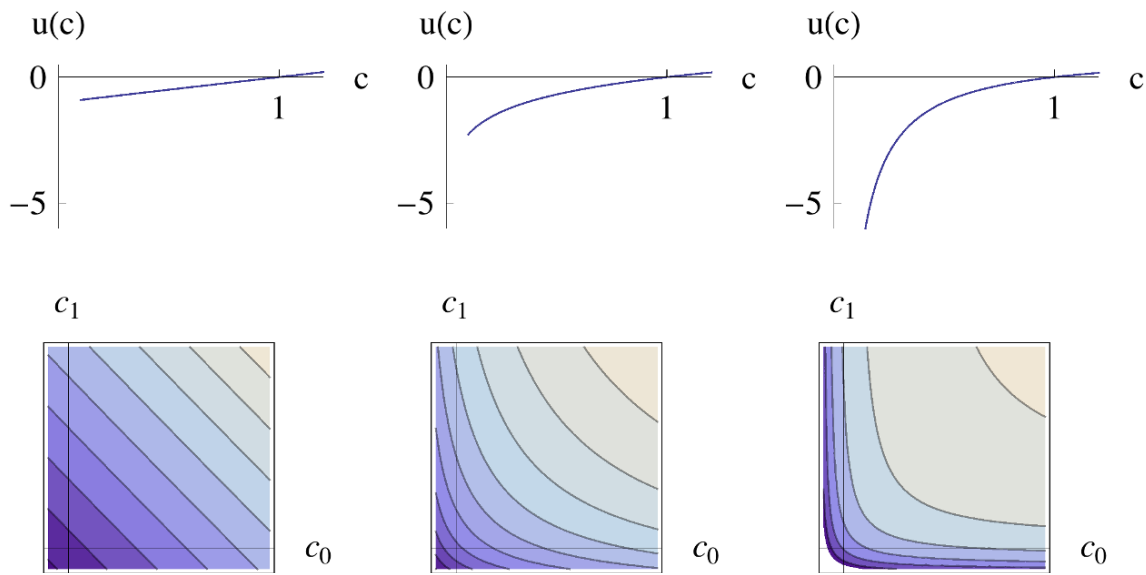


Figure 1.2: CIES preferences: Period utility function and indifference curves for $\sigma = 0.01, 0.99, 2.00$ (from left to right).

The constant elasticity of substitution (CES) production function,

$$f(K, L) = \left(\alpha K^{1-\frac{1}{\theta}} + (1-\alpha)L^{1-\frac{1}{\theta}} \right)^{\frac{\theta}{\theta-1}}, \theta > 0,$$

constitutes a tractable example of a neoclassical production function. The elasticity of substitution between K and L is constant in this case and equals θ . For $\theta \rightarrow \infty$, the CES production function approaches a linear production function, and for $\theta \rightarrow 0$ it approaches the Leontief production function.

For $\theta \rightarrow 1$, the CES production function converges to the Cobb-Douglas production function,

$$f(K, L) = K^\alpha L^{1-\alpha}.$$

When production factors are paid their marginal products then the Cobb-Douglas production function implies constant factor shares, $Kf_K(K, L)/f(K, L) = \alpha$ and $Lf_L(K, L)/f(K, L) = 1 - \alpha$.

1.3 A Short History

In the late 1950s, microeconomic general equilibrium theory with its emphasis on methodological individualism coexisted with Keynesian macroeconomic theory that studied the interplay between economic aggregates, often with a short-run perspective. During the 1960s, empirical macroeconomics suffered setbacks, not least because the Phillips curve proved less stable than expected. Friedman's (1968, p. 8) dictum

that the Phillips curve presumed “a world in which everyone anticipated that nominal prices would be stable and in which that anticipation remained unshaken and immutable whatever happened” stimulated the search for models that better reconciled micro- and macroeconomics (see for example Phelps, 1970). More and more economists found the coexistence of the two fields, and the “neoclassical synthesis” that arose from it unsatisfactory.

By the late 1970s, the profession had lost faith in policy analysis based on large-scale macro econometric models. It had become clear that optimizing behavior also concerns expectation formation, reduced form relationships in models without micro foundations typically are not policy-invariant (Lucas, 1976), and “claims for identification in these [large-scale statistical macroeconomic] models” are unfounded (Sims, 1980, p. 1).

Building on Muth (1961), Lucas, Sargent, and many other macroeconomists adopted the “rational expectations” consistency requirement. Early micro founded dynamic general equilibrium models in the 1980s lacked plausible frictions and abstracted from heterogeneity. Their usefulness for positive and normative analysis was widely debated and many policymakers continued to think in terms of IS-LM-style models without micro foundations, but with ad-hoc frictions deemed empirically relevant.

Since then, the different schools of thought have been converging, thanks to an improved ability to incorporate heterogeneity and frictions of various types into micro founded frameworks. Modern macroeconomic models used for policy analysis typically feature dynamic choices by heterogeneous agents under risk and subject to various frictions; they are referred to as dynamic stochastic general equilibrium (DSGE) models.

1.4 Bibliographic Notes

Arrow and Debreu (1954) prove existence of general equilibrium and Debreu (1959) proves the welfare theorems. Mas-Colell, Whinston and Green (1995) cover microeconomic theory. Debreu (1959, 7) defines commodities with reference to the event tree. Cobb and Douglas (1928) discuss the production function named after them. Blanchard (2000) offers a perspective on the history of macroeconomics during the 20th century.

Chapter 2

Consumption And Saving

The consumption-saving tradeoff of households constitutes the backbone of most modern macroeconomic models. In this chapter, we study the household's dynamic utility maximization problem and the induced demand functions. Throughout, we abstract from risk and assume that leisure does not enter preferences.

2.1 Consumption Smoothing

Consider a household at date t that owns a (possibly negative) stock of assets, a_t . The household receives (or pays) interest income on the assets, $a_t(R_t - 1)$, where R_t denotes the gross interest rate, and receives exogenous wage income, w_t . The stock of assets and the two incomes can be used for consumption, c_t , or to carry assets into the next period, a_{t+1} . The *dynamic budget constraint* reads

$$a_{t+1} = a_t R_t + w_t - c_t.$$

It states that the change in the asset position, $a_{t+1} - a_t$, equals saving, that is income minus consumption.

2.1.1 Two Periods

With two periods, the household's objective function is given by

$$u(c_0) + \beta u(c_1).$$

When the household has no assets to start with ($a_0 = 0$), the dynamic budget constraints at the two dates read

$$\begin{aligned} a_1 &= w_0 - c_0, \\ a_2 &= a_1 R_1 + w_1 - c_1. \end{aligned}$$

Since the household "dies" at the end of period $t = 1$ nobody will lend it resources in that period and terminal household assets therefore must be non-negative, $a_2 \geq 0$.

Moreover, carrying strictly positive assets into period $t = 2$ would be wasteful. The optimal saving choice at date $t = 1$ thus corresponds to $a_2 = 0$.

Using this result and combining the two dynamic budget constraints, we arrive at the *intertemporal budget constraint*

$$c_0 + \frac{c_1}{R_1} = w_0 + \frac{w_1}{R_1}.$$

The terms on the left-hand side represent total spending on the two goods, consumption in the first and second period, c_0 and c_1 respectively. Consumption in the first period is the numeraire, its price is normalized to unity. The relative price of consumption in the second period is given by the inverse of the gross interest rate, $1/R_1$. Intuitively, reducing consumption in the first period by one unit raises saving and increases consumption in the second period by R_1 units. One unit of first-period consumption therefore buys R_1 units of second-period consumption, or one unit of second-period consumption costs $1/R_1$ units of first-period consumption.

The terms on the right-hand side of the intertemporal budget constraint represent the household's wealth, composed of the market values of first- and second-period wage income. Note that the intertemporal budget constraint is isomorphic to the budget constraint in a static model of consumer choice.

To find the household's optimal level of saving in the first period, we may solve the dynamic budget constraints for consumption and substitute the resulting expressions into the objective function. The household's program then reads

$$\max_{a_1} u(w_0 - a_1) + \beta u(a_1 R_1 + w_1).$$

An interior solution to this program satisfies the first-order condition or *Euler equation*

$$u'(c_0) = \beta R_1 u'(c_1) \quad \text{or} \quad \frac{u'(c_0)}{\beta u'(c_1)} = R_1$$

where we re-introduce the variables c_0 and c_1 for ease of notation.

The second representation of the Euler equation states that the marginal rate of substitution between current and future consumption is equated with the relative price between the two goods. This is the same condition as in a static model with "apples" and "oranges" where the price line is tangent to the highest indifference curve.

The Euler equation characterizes optimal second-period consumption relative to first-period consumption and thus, the slope of the optimal consumption path. Three factors determine whether and how strongly consumption increases or decreases over time. First, β . Ceteris paribus, more patience implies a larger ratio $u'(c_0)/u'(c_1)$ and thus (if u is strictly concave), a larger ratio c_1/c_0 . Second, R_1 . A higher interest rate (cheaper second-period consumption) also implies a larger c_1/c_0 . Third, the curvature of the marginal utility function (recall figure 1.2). It determines how strongly a change of β or R_1 translates into a steeper or flatter consumption profile. More curvature implies a stronger *consumption smoothing* motive that is, less willingness to intertemporally substitute.

To solve for the equilibrium consumption levels in terms of the exogenous variables we combine the Euler equation and the intertemporal budget constraint (or the two dynamic budget constraints). If the period utility function is of the CIES type (such that the Euler equation reads $c_0^{-\sigma} = \beta R_1 c_1^{-\sigma}$) this yields

$$c_0 \left(1 + \frac{(\beta R_1)^{1/\sigma}}{R_1} \right) = w_0 + \frac{w_1}{R_1}.$$

From the budget constraint, we may also solve for a_1 and c_1 .

We have completely characterized optimal consumption conditional on $\beta, u, R_1, w_0,$ and w_1 . Two important results emerge. First, optimal consumption depends on lifetime wealth, $w_0 + w_1/R_1$, or *permanent income*, not (only) on contemporaneous income, as with a Keynesian consumption function. This is a consequence of the household's desire to smooth consumption over the life cycle if marginal utility is decreasing ($u'' < 0$) and of the fact that the saving margin allows the household to achieve such smoothing.

Second, a change of interest rate affects optimal consumption threefold, through two *income* or *wealth effects* and one *substitution effect*. First, if $w_1 > 0$, a higher interest rate reduces wealth because it lowers the market value of future labor income expressed in terms of the numeraire, c_0 . This leads the household to consume less in both periods (vice versa for a lower interest rate). Second, for given quantities c_0 and $c_1 > 0$, a higher interest rate lowers the cost of the bundle (c_0, c_1) expressed in terms of the numeraire. The higher interest rate thus raises the household's purchasing power and leads it to consume more of both goods (vice versa for a lower rate). Finally, from the Euler equation, it is optimal to substitute towards the cheaper good. A higher interest rate thus leads the household to increase c_1 relative to c_0 (vice versa for a lower rate). The strength of the substitution effect depends on the intertemporal elasticity of substitution, $1/\sigma$.

Figure 2.1 illustrates the three effects. The black indifference curve, budget line and tangency point characterize the equilibrium at a low interest rate, the red counterparts the equilibrium at a higher interest rate. The substitution effect corresponds to the distance between the gray and the black point, the income effect due to increased purchasing power to the distance between the blue and the gray point, and the wealth effect due to lower discounted labor income to the distance between the red and the blue point. The figure is plotted for the logarithmic utility case, $\sigma = 1$, such that the income effect due to increased purchasing power and the substitution effect on c_0 exactly cancel.

2.1.2 More Periods

More generally, the household's program comprises a multi-period objective function and multiple dynamic budget constraints in addition to the initial and terminal condi-

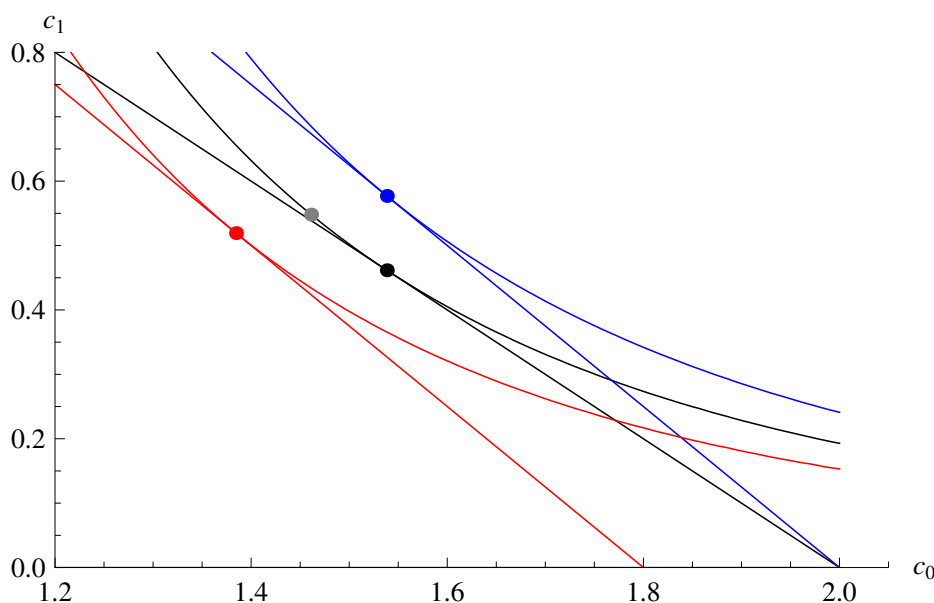


Figure 2.1: Income/wealth and substitution effects due to a change of interest rate. The black (red) tangency point indicates the equilibrium at a low (high) interest rate.

tions:

$$\max_{c_0, \dots, c_T, a_1, \dots, a_{T+1}} \sum_{t=0}^T \beta^t u(c_t) \quad \text{s.t.} \quad a_{t+1} = a_t R_t + w_t - c_t, \quad a_0 R_0 \text{ given}, \quad a_{T+1} \geq 0.$$

In parallel with the strategy adopted in the two-period case, we can confront this problem by solving each of the dynamic budget constraints for consumption and substituting the resulting expressions into the objective function. This yields

$$\max_{a_1, \dots, a_{T+1}} \sum_{t=0}^T \beta^t u(a_t R_t + w_t - a_{t+1}) \quad \text{s.t.} \quad a_0 R_0 \text{ given}, \quad a_{T+1} \geq 0.$$

Differentiating with respect to the choice variables, we find a set of Euler equations.

Lagrangian

An alternative strategy uses the intertemporal budget constraint. The latter can be derived, as before, by combining the dynamic budget constraints:

$$\begin{aligned} a_{T+1} &= a_T R_T + w_T - c_T \\ &= (a_{T-1} R_{T-1} + w_{T-1} - c_{T-1}) R_T + w_T - c_T \\ &= \dots \\ &= a_0 R_0 R_1 \cdots R_T + (w_0 - c_0) R_1 R_2 \cdots R_T + (w_1 - c_1) R_2 \cdots R_T \\ &\quad + \dots + (w_{T-1} - c_{T-1}) R_T + (w_T - c_T). \end{aligned}$$

Let $q_t \equiv (R_1 R_2 \cdots R_t)^{-1}$ denote the price of date- t consumption at the initial date and define $q_0 \equiv 1$. Multiplying the intertemporal budget constraint by q_T and using the optimality condition $a_{T+1} = 0$ yields

$$q_T a_{T+1} = 0 = a_0 R_0 + \sum_{t=0}^T q_t (w_t - c_t) \quad \text{or} \quad \sum_{t=0}^T q_t c_t = a_0 R_0 + \sum_{t=0}^T q_t w_t.$$

In parallel to the two-period case, the intertemporal budget constraint equalizes life-time consumption spending and lifetime wealth at market prices.

Replacing the $T + 1$ dynamic budget constraints with the single intertemporal budget constraint, the household's program can be expressed as

$$\max_{c_0, \dots, c_T} \sum_{t=0}^T \beta^t u(c_t) \quad \text{s.t.} \quad a_0 R_0 + \sum_{t=0}^T q_t (w_t - c_t) = 0.$$

Forming the Lagrangian (see appendix A.1)

$$\mathcal{L} = \sum_{t=0}^T \beta^t u(c_t) + \lambda [a_0 R_0 + \sum_{t=0}^T q_t (w_t - c_t)]$$

and differentiating yields the first-order conditions

$$\beta^t u'(c_t) = \lambda q_t, \quad t = 0, \dots, T.$$

The condition states that marginal utility from consumption in a period equals the price of consumption in that period, q_t , times the multiplier attached to the intertemporal budget constraint, the shadow value of wealth λ .

Combining the first-order conditions at date t and $t + 1$ yields the Euler equation, $u'(c_t) = \beta R_{t+1} u'(c_{t+1})$. In parallel to the two-period case, the Euler equation characterizes the slope of the optimal consumption path. To find the optimal consumption levels, we can combine the Euler equations and the intertemporal budget constraint.

Yet another approach to solving the household's program relies on forming a Lagrangian that incorporates the dynamic budget constraints and the terminal condition $a_{T+1} \geq 0$ rather than the intertemporal budget constraint. This Lagrangian reads (see appendix A.1)

$$\mathcal{L} = \sum_{t=0}^T \{ \beta^t u(c_t) - \lambda_t [a_{t+1} - (a_t R_t + w_t - c_t)] \} + \mu a_{T+1}.$$

The first-order conditions with respect to c_0, \dots, c_T and a_1, \dots, a_T are given by

$$\begin{aligned} \beta^t u'(c_t) &= \lambda_t, \quad t = 0, \dots, T, \\ \lambda_t &= \lambda_{t+1} R_{t+1}, \quad t = 0, \dots, T - 1, \end{aligned}$$

respectively. The first-order condition with respect to a_{T+1} is $\lambda_T = \mu$ and the complementary slackness condition is given by $\mu a_{T+1} = 0$.

Combining the first-order conditions with respect to consumption again yields the Euler equation, $u'(c_t) = \beta R_{t+1} u'(c_{t+1})$. Moreover, non-satiation ($u' > 0$) implies $\lambda_T > 0$ and thus $\mu > 0$ which in turn implies $a_{T+1} = 0$, the *transversality condition* we had informally argued before. The Euler equation characterizes the slope of the optimal consumption path. The Euler equations together with the dynamic budget constraints as well as a_0 and the transversality condition pin down the optimal consumption levels.

Dynamic Programming

Since preferences are time-separable we may also solve the household's program using *dynamic programming* techniques. Recall that an indirect utility function gives the maximal utility as a function of the parameter(s) of the problem. In the context of a consumption saving program at date t , the parameters of the indirect utility function include the level of initial assets, a_t , as well as preference parameters, wages and interest rates over the remaining lifetime.

The household's *value function* at date t is the indirect utility function and the corresponding parameters are referred to as the *state* or state variable(s). Incorporating all exogenous elements of the state into the time subscript, we can express the value function at date t as a function V_t of assets.¹ Note that in contrast to wages and interest rates, a_t is an endogenous state variable: It constitutes a parameter in the program at date t but is a choice variable in earlier periods.

For brevity, let DBC_t denote the dynamic budget constraint at date t and let \mathcal{C} denote the set of dynamic budget constraints at date $t+1$ and later as well as the terminal condition $a_{T+1} \geq 0$. The value function at date t then satisfies

$$V_t(a_t) = \max_{\{c_s, a_{s+1}\}_{s=t}^T} \sum_{s=t}^T \beta^{s-t} u(c_s) \quad \text{s.t. } \text{DBC}_t, \mathcal{C}, a_t \text{ given.}$$

Alternatively, it can be represented recursively as

$$V_t(a_t) = \max_{c_t, a_{t+1}} \{u(c_t) + \beta V_{t+1}(a_{t+1})\} \quad \text{s.t. } \text{DBC}_t, a_t \text{ given.}$$

To see this, simply rearrange terms:

$$\begin{aligned} V_t(a_t) &= \max_{\{c_s, a_{s+1}\}_{s=t}^T} \sum_{s=t}^T \beta^{s-t} u(c_s) \quad \text{s.t. } \text{DBC}_t, \mathcal{C}, a_t \text{ given} \\ &= \max_{c_t, a_{t+1}} u(c_t) + \left(\max_{\{c_s, a_{s+1}\}_{s=t+1}^T} \sum_{s=t+1}^T \beta^{s-t} u(c_s) \quad \text{s.t. } \mathcal{C}, a_{t+1} \text{ given} \right) \quad \text{s.t. } \text{DBC}_t, a_t \text{ given} \\ &= \max_{c_t, a_{t+1}} u(c_t) + \beta \left(\max_{\{c_s, a_{s+1}\}_{s=t+1}^T} \sum_{s=t+1}^T \beta^{s-(t+1)} u(c_s) \quad \text{s.t. } \mathcal{C}, a_{t+1} \text{ given} \right) \quad \text{s.t. } \text{DBC}_t, a_t \text{ given} \\ &= \max_{c_t, a_{t+1}} u(c_t) + \beta V_{t+1}(a_{t+1}) \quad \text{s.t. } \text{DBC}_t, a_t \text{ given.} \end{aligned}$$

¹Typically, the set of state variables is not unique. For example, the state may be defined to include t or alternatively $T - t$.

We are confronted with a system of functional equations often referred to as *Bellman equations*. These functional equations stipulate equality of functions (rather than functions evaluated at certain points). Substituting the dynamic budget constraint yields a compact representation of the Bellman equation,

$$V_t(a_t) = \max_{a_{t+1}} u(a_t R_t + w_t - a_{t+1}) + \beta V_{t+1}(a_{t+1}),$$

which has to hold at all dates and for all feasible values of a_t .

Since T is finite we can solve the system of Bellman equations by backward induction. To start the induction, note that $V_{T+1}(a_{T+1}) = 0$ for all a_{T+1} . At date T , this implies the optimal choice $a_{T+1} = 0$ and thus, the value function $V_T(a_T) = u(a_T R_T + w_T)$. Using this result and the Bellman equation at date $T - 1$ we can characterize the optimal choice and the value function at date $T - 1$. Proceeding backward, we can solve for all value functions V_t and *policy functions* g_t say; the latter give the optimal value of the choice variable as a function of the state, $a_{t+1} = g_t(a_t)$. It is straightforward to write a computer program that iteratively computes approximations of V_t and g_t for $t = T, T - 1, T - 2, \dots$

To derive the Euler equation, we do not need to know the functional form of the value functions. As long as regularity conditions are satisfied it suffices to use the first-order and envelope conditions,

$$\begin{aligned} u'(c_t) &= \beta V'_{t+1}(a_{t+1}), \\ V'_t(a_t) &= u'(c_t)(R_t - g'_t(a_t)) + \beta V'_{t+1}(a_{t+1})g'_t(a_t) \\ &= u'(c_t)R_t. \end{aligned}$$

The first-order condition in the first line results from differentiating the right-hand side of the Bellman equation with respect to the choice variable. An optimal choice of a_{t+1} assures that this condition is satisfied. The envelope condition in the second line results from differentiating the Bellman equation with respect to the state variable. The last equality follows from substituting the first-order condition into the second condition—this is the envelope theorem at work (see below). Combining the first-order condition and the envelope condition (evaluated at $t + 1$) yields the Euler equation, $u'(c_t) = \beta R_{t+1} u'(c_{t+1})$. It can also be expressed as the functional equation

$$u'(a_t R_t + w_t - g_t(a_t)) = \beta R_{t+1} u'(g_t(a_t) R_{t+1} + w_{t+1} - g_{t+1}(g_t(a_t)))$$

which must hold for all feasible values of a_t .

Figure 2.2 illustrates the envelope condition at date $T - 1$. The figure plots the maximand on the right-hand side of the Bellman equation at date $T - 1$, $\text{RHS}(a_T; a_{T-1}) \equiv u(a_{T-1} R_{T-1} + w_{T-1} - a_T) + \beta u(a_T R_T + w_T)$, against a_{T-1} (the endogenous state) and a_T (the choice variable). For a given value of the state, the maximand is a concave function of the choice variable that reaches a maximum at the optimal choice. Starting at a given value of the state and the corresponding optimal choice, a small increase in the state has two effects. First, it directly alters the maximand, corresponding to the change of RHS for a move to the right in parallel to the a_{T-1} axis. Second, it induces an

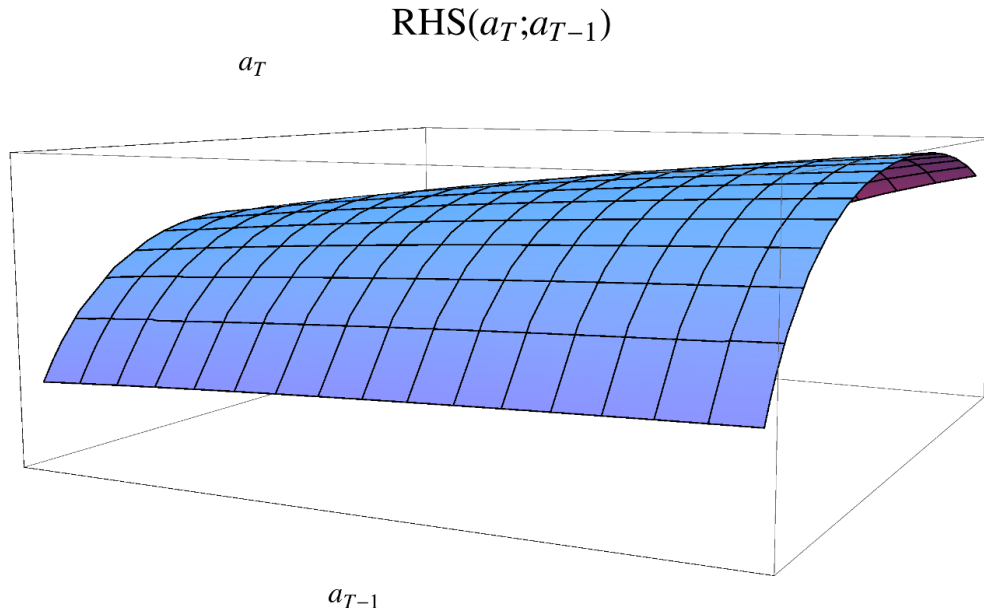


Figure 2.2: Envelope condition: $\text{RHS}(a_T; a_{T-1})$ plotted against a_{T-1} and a_T .

adjustment of the optimal choice, a_T . But since the derivative of the maximand with respect to a_T was zero to start with, the effect of this induced change on RHS is of second order: $u'(c_t)(-g'_t(a_t)) + \beta V'_{t+1}(a_{t+1})g'_t(a_t) = 0$. The only first-order effect of an infinitesimal change of state on the value function thus is the direct one.

2.1.3 Infinite Horizon

There are several reasons to consider optimization of households (and other agents) over an infinite horizon, $T \rightarrow \infty$. First, because this can be interpreted as reflecting intergenerational altruism: Parents care about the utility of their children who in turn care about the utility of their children, and so on. Second, because it can be interpreted as reflecting a time invariant survival probability. And third, because eliminating time as a state variable makes the program simpler.

To derive the household's intertemporal budget constraint in the infinite-horizon case, we need to specify the terminal condition. One might think that the constraint in the finite-horizon case, $a_{T+1} \geq 0$, generalizes to $\lim_{T \rightarrow \infty} a_{T+1} \geq 0$. But the latter constraint would be unnecessarily tight. Instead, the appropriate constraint is given by the "no-Ponzi-game condition", $\lim_{T \rightarrow \infty} q_T a_{T+1} \geq 0$. Stated differently, the constraint $q_T a_{T+1} \geq 0$ rather than the constraint $a_{T+1} \geq 0$ generalizes from the finite to the infinite-horizon case.

The unnecessarily tight condition $\lim_{T \rightarrow \infty} a_{T+1} \geq 0$ would prevent the household from holding debt in the long run. In contrast, the no-Ponzi-game condition only rules out debt positions that grow at a rate weakly higher than the gross interest rate. The household thus is prevented from permanently rolling over debt, including interest,

and never servicing it. Satisfying the no-Ponzi-game condition guarantees that going forward, the present value of debt service fully covers the outstanding debt. The time profile of debt service may take various forms. For example, the debtor may once and for all pay back all outstanding debt. Or the debtor may never repay the principal but forever pay net interest on the outstanding liabilities.

Using the no-Ponzi-game condition and following similar steps as in the finite-horizon case, we can derive the infinite-horizon intertemporal budget constraint

$$a_0R_0 + \lim_{T \rightarrow \infty} \sum_{t=0}^T q_t(w_t - c_t) = \lim_{T \rightarrow \infty} q_T a_{T+1} \geq 0.$$

By the same logic as in the finite-horizon case, optimality requires setting $\lim_{T \rightarrow \infty} q_T a_{T+1}$ as small as possible (see appendix B.1). The intertemporal budget constraint therefore reduces to $a_0R_0 + \sum_{t=0}^{\infty} q_t(w_t - c_t) = 0$ and the household's program can be represented as

$$\max_{\{c_t\}_{t \geq 0}} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{s.t.} \quad a_0R_0 + \sum_{t=0}^{\infty} q_t(w_t - c_t) = 0.$$

Forming the Lagrangian

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \lambda [a_0R_0 + \sum_{t=0}^{\infty} q_t(w_t - c_t)]$$

and differentiating yields the same first-order conditions as before,

$$\beta^t u'(c_t) = \lambda q_t,$$

and thus, the Euler equation.

Turn next to dynamic programming. With an infinite horizon, the household's horizon always is the same, independently of how many periods have gone by. The structure of the optimization problem therefore is independent of time—the problem is *time autonomous*—unless wages, interest rates, or preferences are time dependent. In the time autonomous case, the Bellman equation can be expressed as

$$V(a_0) = \max_{a_+} u(a_0R + w - a_+) + \beta V(a_+).$$

Note that the value functions on the left- and right-hand side are identical (the functions do not have time subscripts), in contrast with the finite-horizon case. The state variable a_0 and the choice variable a_+ are written without time subscripts to indicate the time autonomous nature of the program.

Although in the infinite-horizon case no final period exists, one can nevertheless find the value function V of the infinite-horizon problem by means of an iterative procedure that parallels the solution strategy in the finite-horizon case. This follows from mathematical results which establish that under certain conditions, (i) the value function V solving the time-autonomous Bellman equation is unique, and (ii) starting from any value function guess (for example the function that started the recursion in the finite-horizon case, $V_{T+1}(a_{T+1}) = 0$), the iterative procedure applied in the

finite-horizon case yields a sequence of value functions that converges to V (see appendix A.2). When working with a computer, an approximation of the infinite-horizon value function thus can be found by running exactly the same code as in the finite horizon case except that the iterative procedure only is stopped when the sequence of value function approximations has converged.

2.2 Extensions

2.2.1 Borrowing Constraint

We have assumed that households may freely borrow as long as they satisfy the intertemporal budget constraint that is, as long as they are solvent. But borrowing against future wage income may be difficult, for example because a potential lender does not have sufficient information about the future income stream or cannot enforce repayment. This renders the future wage income illiquid and imposes a new constraint—a liquidity or borrowing constraint—in addition to the intertemporal budget constraint.

The simplest possible borrowing constraint excludes all borrowing against future wage income. The household's financial assets then must be positive at all times, $a_{t+1} \geq 0$. A binding borrowing constraint is costly because it prevents consumption smoothing. To see this, consider a two-period setting with strictly concave preferences and suppose that absent a borrowing constraint, optimal consumption in the first period exceeds "cash at hand," $w_0 + a_0R_0$, and thus requires setting $a_1 < 0$. The borrowing constraint renders this plan infeasible. Constrained optimal consumption then equals $(c_0, c_1) = (w_0 + a_0R_0, w_1)$ that is, consumption follows income, as with a Keynesian consumption function. Note that we have identified a new saving motive: reduced borrowing due to a binding borrowing constraint.

More formally, the Lagrangian associated with the constrained saving problem reads

$$\mathcal{L} = u(c_0) + \beta u(c_1) - \lambda \left(c_0 + \frac{c_1}{R_1} - w_0 - \frac{w_1}{R_1} - a_0R_0 \right) + \mu(w_0 + a_0R_0 - c_0),$$

where the non-negative multiplier μ represents the shadow cost of the borrowing constraint $w_0 + a_0R_0 - c_0 \geq 0$. Differentiating with respect to c_0 and c_1 and combining the two conditions yields the modified Euler equation

$$u'(c_0) = \beta R_1 u'(c_1) + \mu.$$

A binding borrowing constraint, $\mu > 0$, increases the slope of the equilibrium consumption path. Moreover, $\mu > 0$ and the complementary slackness condition, $\mu(w_0 + a_0R_0 - c_0) = 0$, imply $c_0 = w_0 + a_0R_0$, in line with the heuristic argument above.

2.2.2 Non-Geometric Discounting and Time-Consistency

Until now, we have posited that the sequence of psychological discount factors is geometrically declining, $1, \beta, \beta^2, \beta^3, \dots$. Under this assumption, a household that re-optimizes period by period opts to continue with the consumption plan chosen earlier in time. That is, if the household optimally chose the consumption plan (c_t, \vec{c}_{t+1}) with $\vec{c}_{t+1} \equiv \{c_{t+1}, c_{t+2}, \dots\}$ at date t , then pursuing the plan \vec{c}_{t+1} once time has progressed to date $t + 1$ remains optimal. As a consequence, it does not matter whether we assume that the household chooses the consumption plan at date $t = 0$ once and for all, under *commitment*, or whether it re-optimizes period by period. In this sense, the initial consumption plan is *time-consistent*.

Under more general assumptions about the psychological discount factor sequence the consumption plan at date $t = 0$ need not be time-consistent. Two households, one re-optimizing period by period and the other acting under commitment, may end up with different consumption paths although their preferences at date $t = 0$ and their budget sets are identical.

Consider an extreme example in a three-period setting where the household discounts the immediate future at factor β but does not additionally discount payoffs in subsequent periods. At date $t = 0$, the household has preferences $u(c_0) + \beta(u(c_1) + u(c_2))$ while at date $t = 1$, preferences are given by $u(c_1) + \beta u(c_2)$. For simplicity, let $R_t = 1, w_t = w$. The optimal consumption path as of date $t = 0$ then solves the problem

$$\max_{c_0, c_1, c_2} u(c_0) + \beta(u(c_1) + u(c_2)) \quad \text{s.t.} \quad c_0 + c_1 + c_2 = 3w + a_0,$$

which yields $c_1 = c_2$ (assuming $u'' < 0$). A household who can commit implements this solution.

In contrast, for given a_1 the optimal consumption path as of date $t = 1$ solves

$$\max_{c_1, c_2} u(c_1) + \beta u(c_2) \quad \text{s.t.} \quad c_1 + c_2 = 2w + a_1,$$

which yields $c_1 > c_2$. Absent commitment, a household re-optimizing at date $t = 1$ thus does not implement the path that is optimal at date $t = 0$. In other words, without commitment, the ex-ante optimal consumption plan cannot be implemented.

In equilibrium without commitment, the two “selves” of the household play a game against each other. The first self chooses c_0 and a_1 . The second self chooses c_1, a_2 and c_2 conditional on a_1 . Since the second self chooses $c_1 > c_2$, the first self cannot implement the ex-ante optimal plan. Anticipating the second self’s “distorted” action, the first self solves

$$\max_{c_0, c_1, c_2} u(c_0) + \beta(u(c_1) + u(c_2)) \quad \text{s.t.} \quad c_0 + c_1 + c_2 = 3w + a_0, u'(c_1) = \beta u'(c_2)$$

where the second constraint, the Euler equation of the second self, reflects the “distorted” consumption choice from date $t = 1$ onward. By choosing a_1 the first self affects the state variable at date $t = 1$ and may thus influence the action taken by the second self.

2.2.3 Multiple Goods

Consider a two-period lived household that consumes two goods in each period. Their quantities, d_t and e_t , are aggregated into a CES consumption index,

$$c_t(d_t, e_t) = \left(\delta^{\frac{1}{\theta}} d_t^{1-\frac{1}{\theta}} + \varepsilon^{\frac{1}{\theta}} e_t^{1-\frac{1}{\theta}} \right)^{\frac{\theta}{\theta-1}}, \quad \delta + \varepsilon = 1, \quad \theta > 0,$$

with elasticity of substitution equal to θ . The price of d_t is normalized to unity and the relative price of e_t is denoted p_t . Intertemporal preferences are described by the utility function $u(c_0) + \beta u(c_1)$.

The household's consumption choice has an intratemporal dimension (the trade-off between d_t and e_t) and an intertemporal one (the trade-off between c_0 and c_1). Focusing first on the intratemporal trade-off, consider the problem of maximizing c_t subject to a given amount of spending, $z_t = d_t + p_t e_t$. The solution to this problem is given by

$$d_t = \delta \frac{z_t}{\delta + \varepsilon p_t^{1-\theta}}, \quad e_t = \varepsilon p_t^{-\theta} \frac{z_t}{\delta + \varepsilon p_t^{1-\theta}}, \quad c_t = \left(\delta + \varepsilon p_t^{1-\theta} \right)^{\frac{1}{\theta-1}} z_t.$$

Solving the latter equation for z_t we can derive a price index, \mathcal{P}_t : One unit of the consumption index c_t costs

$$\mathcal{P}_t = \left(\delta + \varepsilon p_t^{1-\theta} \right)^{\frac{1}{1-\theta}}.$$

For $p_t = 1$, the price index equals unity. For $p_t \rightarrow \infty$, it increases with p_t if $\theta < 1$ but converges to a constant if $\theta > 1$. Intuitively, how strongly the relative price increase translates into a higher price index depends on the household's willingness to substitute across goods. Using the price index, we also have $d_t = \delta c_t \mathcal{P}_t^\theta$ and $e_t = \varepsilon c_t (\mathcal{P}_t / p_t)^\theta$.

Equipped with these results, we turn to the intertemporal program. The dynamic budget constraints are given by $w_0 = \mathcal{P}_0 c_0 + a_1$ and $a_1 R_1 + w_1 = \mathcal{P}_1 c_1$ where wage income and assets are expressed in terms of the numeraire d_t . The household's program therefore reads

$$\max_{c_0, c_1} u(c_0) + \beta u(c_1) \quad \text{s.t.} \quad \mathcal{P}_0 c_0 + \frac{\mathcal{P}_1 c_1}{R_1} = w_0 + \frac{w_1}{R_1}$$

and the Euler equation characterizing the optimal intertemporal consumption allocation is given by

$$u'(c_0) = \beta R_1 \frac{\mathcal{P}_0}{\mathcal{P}_1} u'(c_1).$$

As usual, the marginal rate of substitution is equated with the marginal rate of transformation. But with a consumption index, the marginal rate of transformation is given by the *own rate of interest*, $R_1 \mathcal{P}_0 / \mathcal{P}_1$. The latter differs from the marginal rate of transformation for the numeraire good, R_1 , whenever \mathcal{P}_t (and thus, p_t) changes over time.

2.3 Bibliographic Notes

Modigliani and Brumberg (1954) and Friedman (1957) derive consumption functions based on microeconomic reasoning. Friedman (1957) refers to permanent income. Modigliani and Brumberg (1954) stress life cycle considerations, among others. Strotz (1956) analyzes time inconsistency and Laibson (1997) explores the consequences of hyperbolic rather than geometric discounting.

Chapter 3

Dynamic Competitive Equilibrium

We now embed the consumption-saving tradeoff of households in two general equilibrium models of capital accumulation: We add a firm sector and impose market clearing. In the first model, the *representative agent* or “*Ramsey*” model, we assume that households are homogeneous. This is a convenient, but strong assumption; appendix B.2 provides some discussion. In the second model, the *overlapping generations* model, we consider the interaction between households of different age. Throughout the chapter, we abstract from risk and assume that leisure does not enter preferences such that labor supply is exogenous.

3.1 Homogeneous Households And Capital Accumulation

3.1.1 Economy

The economy is inhabited by a continuum of identical households of mass one as well as a continuum of firms. Being alike, the households act in many respects like a single, representative household. However, since they are “small,” they take prices as given. Similarly, firms are identical and “small” and take prices as given.

3.1.2 Firms

Firms solve static profit maximization problems. In each period, they rent capital K_t at rental rate r_t and labor L_t at wage w_t to produce output with a CRTS production function, f . Profits are distributed to households. The output good is the numeraire.

Taking rental rates as given, the representative firm maximizes

$$\max_{K_t, L_t} f(K_t, L_t) - K_t r_t - L_t w_t.$$

The first-order conditions

$$f_K(K_t, L_t) = r_t, \quad (3.1)$$

$$f_L(K_t, L_t) = w_t \quad (3.2)$$

define demand functions for capital and labor. The budget constraint of the representative firm reads

$$f(K_t, L_t) = K_t r_t + L_t w_t + \pi_t \quad (3.3)$$

where π_t denotes profits. Equilibrium profits equal zero, due to CRTS and price taking.

3.1.3 Households

The representative household maximizes $\sum_{t=0}^{\infty} \beta^t u(c_t)$. The dynamic budget constraint and Euler equation, respectively, are given by

$$\begin{aligned} a_{t+1} &= a_t R_t + w_t - c_t + \pi_t, \\ u'(c_t) &= \beta R_{t+1} u'(c_{t+1}). \end{aligned}$$

Since all households are alike (and the economy is closed and there is no government sector) households do not hold claims vis-a-vis each other or third parties. Accordingly, household assets correspond to the physical capital stock in the economy: The capital stock per worker, k_t , equals a_t . Capital depreciates at rate δ per period. The gross return on household saving thus equals the rental rate on capital paid by firms, r_t , plus the unit of capital net of depreciation: $R_t = 1 + r_t - \delta$.

Combining these conditions yields

$$k_{t+1} = k_t(1 + r_t - \delta) + w_t - c_t + \pi_t, \quad (3.4)$$

$$u'(c_t) = \beta(1 + r_{t+1} - \delta)u'(c_{t+1}). \quad (3.5)$$

Households also satisfy the transversality condition, $\lim_{T \rightarrow \infty} \beta^T u'(c_T) k_{T+1} = 0$.¹

3.1.4 Market Clearing

There are three goods in each period: Labor, capital (inherited from the last period), and output which can be used for consumption and investment (accumulation of new capital). Since there is one representative household whose time endowment per period equals unity, labor and capital market clearing requires firms to demand one unit of labor and k_t units of capital:

$$K_t = k_t, \quad (3.6)$$

$$L_t = 1. \quad (3.7)$$

¹When the horizon is finite, the transversality condition applies without the limit, $k_{T+1} = 0$.

By Walras' Law, market clearing in all but one market implies that the remaining market clears as well if all agents satisfy their budget constraints. In the economy considered here, this can be seen by combining (3.3), (3.4), (3.6), and (3.7) to find

$$k_{t+1} = k_t(1 + r_t - \delta) + w_t - c_t + f(k_t, 1) - k_t r_t - 1w_t$$

which simplifies to the *resource constraint*

$$k_{t+1} = k_t(1 - \delta) + f(k_t, 1) - c_t.$$

The resource constraint states that the market for the output good clears. Equivalently, gross investment plus consumption equals output, or saving equals net investment. The condition represents the GDP identity in a closed economy without government sector.

3.1.5 General Equilibrium

In general equilibrium, the transversality condition as well as conditions (3.1)–(3.7) (and thus, the resource constraint) hold at all dates. The equilibrium conditions can be reduced to (i) the transversality condition, (ii) two core equations that represent the laws of motion for capital and consumption, and (iii) five remaining conditions that determine r_t , w_t , π_t , K_t , and L_t . The two core equations are given by the resource constraint and the Euler equation with the rental rate of capital expressed in terms of the marginal product of capital:

$$k_{t+1} = k_t(1 - \delta) + f(k_t, 1) - c_t, \quad (3.8)$$

$$u'(c_t) = \beta(1 + f_K(k_{t+1}, 1) - \delta)u'(c_{t+1}). \quad (3.9)$$

Note that, conditional on k_t and c_t , these two equations pin down k_{t+1} and c_{t+1} .

For a given initial capital stock, k_0 , conditions (3.8) and (3.9) completely pin down the equilibrium sequences for capital and consumption once a starting value for consumption, c_0 , is specified. This starting value is not "free," however, because the sequences also must satisfy the transversality condition,

$$\lim_{T \rightarrow \infty} \beta^T u'(c_T) k_{T+1} = 0.$$

As we will see below, there is a unique c_0 such that the paths implied by (k_0, c_0) as well as (3.8) and (3.9) also satisfy the transversality condition.

3.1.6 Social Planner Allocation and Pareto Optimality

We have characterized the equilibrium conditions in the *decentralized* economy with firms and households. Alternatively, we can characterize the social planner allocation

in a “Robinson Crusoe” economy. This economy is inhabited by a single consumer-producer who operates the production function f and saves in the form of capital. Robinson Crusoe solves

$$\max_{\{c_t, k_{t+1}\}_{t \geq 0}} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{s.t.} \quad k_{t+1} = k_t(1 - \delta) + f(k_t, 1) - c_t, \quad k_0 \text{ given, } k_{t+1} \geq 0.$$

The non-negativity constraint on capital is not binding if f satisfies the Inada conditions. Solving this program yields exactly the same conditions as those characterizing the decentralized equilibrium, namely conditions (3.8) and (3.9) and the transversality condition (see appendix B.3).

Since the social planner allocation is the feasible allocation preferred by the representative household it necessarily is Pareto optimal. By implication, the decentralized equilibrium allocation is Pareto optimal as well. This is not surprising since the economy satisfies the conditions of the first welfare theorem.

3.1.7 Analysis

Since (3.8) and (3.9) constitute non-linear first-order difference equations the model cannot be solved in closed form. However, we may characterize equilibrium by means of a phase diagram which is constructed based on the relations

$$c_t = f(k_t, 1) - \delta k_t, \quad (3.10)$$

$$1 = \beta(1 + f_K(k_{t+1}, 1) - \delta). \quad (3.11)$$

Condition (3.10) follows from (3.8) when the capital stock is constant over time, $k_t = k_{t+1}$. In this case, consumption plus replacement investment equals output. Condition (3.11) follows from (3.9) when $c_t = c_{t+1}$. For consumption to be constant over time, βR_{t+1} must equal unity.

In figure 3.1, relations (3.10) and (3.11) are represented by the black concave schedule and the black vertical line, respectively. Their intersection defines the steady state of the system, $(k^{\text{mgr}}, c^{\text{mgr}})$, where all equilibrium conditions are satisfied and all variables do not change over time.

From (3.10), consumption is maximized subject to a time invariant capital stock when the latter equals the “golden-rule” capital stock, k^{gr} , which satisfies

$$f_K(k^{\text{gr}}, 1) = \delta.$$

The steady-state or “modified-golden-rule” capital stock is lower than the golden-rule capital stock, $k^{\text{mgr}} < k^{\text{gr}}$, because from (3.11)

$$f_K(k^{\text{mgr}}, 1) = \delta + \beta^{-1} - 1 > \delta.$$

To characterize capital stock dynamics outside of steady state, suppose that $c_t > f(k_t, 1) - \delta k_t$ that is, c_t lies above the concave schedule. Investment then falls short

of the replacement investment necessary to maintain the capital stock, and as a consequence $k_{t+1} < k_t$. Conversely, a choice of c_t below the concave schedule implies $k_{t+1} > k_t$. Consumption dynamics outside of steady state are determined by the Euler equation. If the capital stock is smaller than k^{mgr} then the marginal product of capital and thus, the interest rate are higher than in steady state and consumption rises, $c_{t+1} > c_t$. Conversely, a capital stock larger than k^{mgr} implies $c_{t+1} < c_t$.

The system dynamics therefore differ across the four areas separated by (3.10) and (3.11): If $k_t < k^{\text{mgr}}$ and $c_t < f(k_t, 1) - \delta k_t$ then both the capital stock and consumption rise over time. If $k_t > k^{\text{mgr}}$ and $c_t < f(k_t, 1) - \delta k_t$ then the capital stock rises and consumption falls. If $k_t < k^{\text{mgr}}$ and $c_t > f(k_t, 1) - \delta k_t$ then the capital stock falls and consumption rises. Finally, if $k_t > k^{\text{mgr}}$ and $c_t > f(k_t, 1) - \delta k_t$ then both the capital stock and consumption fall.

The paths indicated by dots in figure 3.1 illustrate the system dynamics. Consider a low initial capital stock, $k_0 = 0.5k^{\text{mgr}}$ say. The figure illustrates three candidate adjustment paths that are distinguished by the initial consumption level, c_0 . All these candidate paths satisfy (3.8) and (3.9) for a while but only one—the blue path—satisfies (3.8) and (3.9) forever and also meets the transversality condition. Too low an initial consumption level implies non-convergent dynamics to the “bottom right” (in red) where the interest rate is negative and thus, the transversality condition violated. Too high an initial consumption level implies non-convergent dynamics to the “top left” (in red) where the Euler equation prescribes consumption growth but household assets tend to zero. Only an intermediate initial consumption value implies convergent dynamics (in blue) towards the steady state, and only this path satisfies (3.8), (3.9), and the transversality condition throughout the transition.

Similarly, for a high initial capital stock, $k_0 = k^{\text{gr}}$ say, too low or too high an initial consumption value implies non-convergent dynamics, indicated by the red paths starting above k^{gr} , while an appropriate intermediate starting value implies convergent dynamics, indicated by the blue path.

As a function of the initial capital stock, k_0 , the associated consumption level guaranteeing convergent equilibrium dynamics, $c_0(k_0)$, traces out the *saddle path*. The saddle path gives the equilibrium initial consumption level for an initial capital stock, and it indicates the path along which convergent equilibrium dynamics occur. The blue dots in figure 3.1 illustrate the segment of the saddle path between $0.5k^{\text{mgr}}$ and k^{gr} .

Beyond the phase diagram, several strategies may be used to solve the model. One, described below, is based on a linear approximation of the difference equation system (3.8) and (3.9); it involves eigenvalue and eigenvector operations. Another strategy is based on simply trying different starting values c_0 conditional on k_0 and checking whether the induced system dynamics are convergent. Finally, one may solve the social planner’s program numerically, using dynamic programming methods.

The approximation strategy rests on linearizing (3.8) and (3.9) about the steady state. For example, totally differentiating (3.8) and evaluating at the steady state yields

$$\begin{aligned} dk_{t+1} + dc_t &= (f_K(k, 1) + 1 - \delta)dk_t, \\ \Rightarrow \hat{k}_{t+1} + \hat{c}_t \frac{c}{k} &= \beta^{-1}\hat{k}_t, \end{aligned}$$

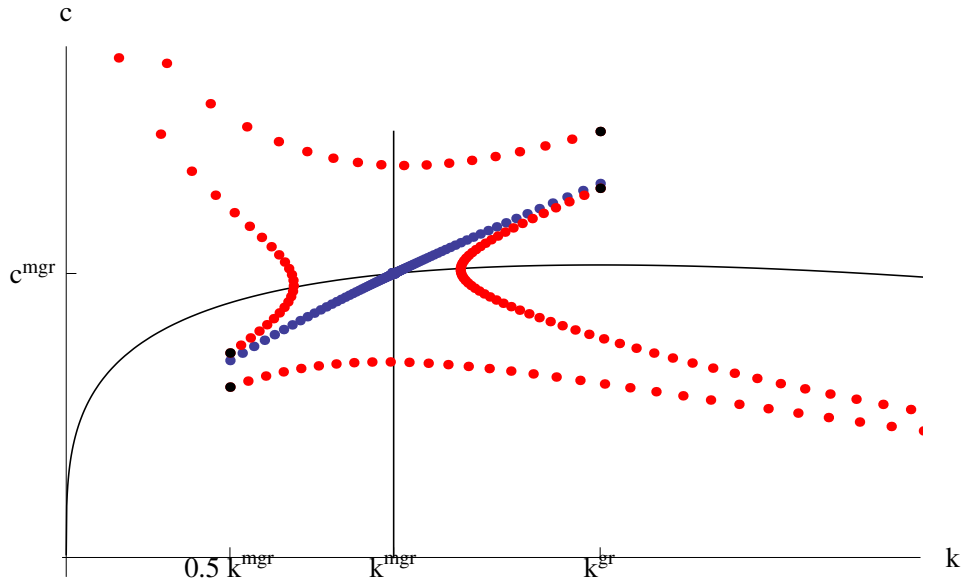


Figure 3.1: Dynamics in the representative agent model: Steady-state resource constraint and Euler equation (in black) as well as (k_t, c_t) -paths for different values of (k_0, c_0) (in red or blue).

where a circumflex denotes infinitesimal relative deviations from the corresponding steady state value, e.g., $\hat{c}_t \equiv (c_t - c)/c$. Similarly, taking logarithms in (3.9), totally differentiating and evaluating at the steady state yields (letting $\sigma \equiv -u''(c)c/u'(c)$)

$$\begin{aligned} \ln(u'(c_t)) &= \ln(\beta) + \ln(1 + f_K(k_{t+1}, 1) - \delta) + \ln(u'(c_{t+1})), \\ \Rightarrow -\sigma \frac{dc_t}{c} &= \frac{f_{KK}(k, 1) dk_{t+1}}{1 + f_K(k, 1) - \delta} - \sigma \frac{dc_{t+1}}{c}, \\ \Rightarrow \hat{c}_t &= -\frac{\beta}{\sigma} f_{KK}(k, 1) k \hat{k}_{t+1} + \hat{c}_{t+1}. \end{aligned}$$

Approximating the original system to the first order means that we apply the linearized system to deviations from steady state even if they are larger than infinitesimal.

Next, we collect the linearized equations:

$$\begin{bmatrix} \hat{c}_{t+1} \\ \hat{k}_{t+1} \end{bmatrix} = M \begin{bmatrix} \hat{c}_t \\ \hat{k}_t \end{bmatrix}, \quad M \equiv \begin{bmatrix} 1 & -\frac{\beta}{\sigma} f_{KK}(k, 1) k \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ -\frac{c}{k} & \beta^{-1} \end{bmatrix} = \begin{bmatrix} 1 - \frac{c}{\sigma} \beta f_{KK} & \frac{f_{KK} k}{\sigma} \\ -\frac{c}{k} & \beta^{-1} \end{bmatrix}.$$

Then we compute the eigenvalues ρ_1 and ρ_2 as well as the corresponding eigenvectors v_1 and v_2 of the matrix M . An eigenvalue of M satisfies $\det(M - \rho I) = 0$ that is, it solves the characteristic equation $\mathcal{C}(\rho) = 0$ with $\mathcal{C}(\rho) \equiv \rho^2 - \rho(1 + \beta^{-1} - c\beta/\sigma f_{KK}(k, 1)) + \beta^{-1}$. The latter is a continuous quadratic function satisfying $\mathcal{C}(0) > 0$, $\mathcal{C}(1) < 0$ and $\lim_{\rho \rightarrow \infty} \mathcal{C}(\rho) = \infty$. It follows that $0 < \rho_1 < 1 < \rho_2$. In fact, $\rho_2 = 1/(\beta\rho_1)$, which holds more generally.

Finally, we use standard results (see appendix A.3) to express the linearized system as

$$\begin{bmatrix} \hat{c}_t \\ \hat{k}_t \end{bmatrix} = \varphi_1 \rho_1^t v_1 + \varphi_2 \rho_2^t v_2,$$

where φ_1, φ_2 are arbitrary constants that remain to be determined. The requirement that system dynamics are stable implies that φ_2 must equal zero since ρ_2^t grows without bound for $t \rightarrow \infty$. The second constant, φ_1 , is pinned down by the initial condition for the capital stock, $\hat{k}_0 = \varphi_1 v_{1[2]}$, where $v_{1[2]}$ denotes the second element of the eigenvector v_1 .

The saddle path of the linearized system is given by the function

$$\hat{c}_0(\hat{k}_0) = \frac{\hat{k}_0}{v_{1[2]}} v_{1[1]}$$

or equivalently,

$$\frac{dc_0}{dk_0} = \frac{c}{k} \frac{v_{1[1]}}{v_{1[2]}}$$

For $\sigma \rightarrow \infty$ or $f_{KK}k/f_K \rightarrow 0$, the latter expression approaches $(1 - \beta)/\beta$. Lower values of σ (a higher intertemporal elasticity of substitution) or more negative elasticities of the marginal product with respect to k increase dc_0/dk_0 .

The speed of convergence of the linearized system is determined by the stable eigenvalue, ρ_1 . The higher this eigenvalue, the slower the rate of convergence towards the steady state.

3.1.8 Population Growth

Suppose the number of household members grows at gross rate ν per period and the household's objective thus equals $\sum_{t=0}^{\infty} \beta^t \nu^t u(c_t)$ where c_t denotes per-capita consumption as before. The resource constraint in per-capita terms now is given by

$$\nu k_{t+1} = k_t(1 - \delta) + f(k_t, 1) - c_t.$$

Population growth implies that the capital-labor ratio at date $t + 1$ is smaller than the per-capita resources not consumed at date t . Except for this difference, the conditions characterizing the centralized or decentralized equilibrium are not affected by population growth.

3.2 Overlapping Generations And Capital Accumulation

3.2.1 Economy

Households live for two periods rather than over an infinite horizon. In each period, a continuum of young and old households of mass one each inhabit the economy, see

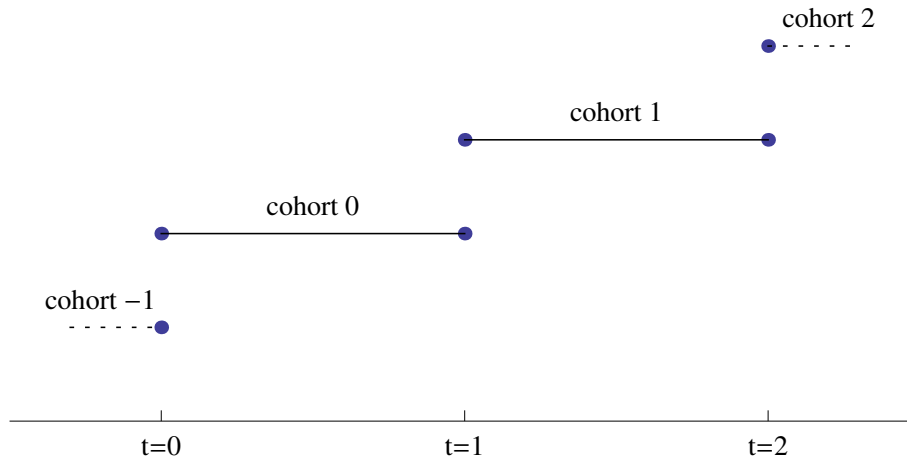


Figure 3.2: Overlapping generations.

figure 3.2. Young households are born without assets. They supply labor, consume, and save for retirement. When old, they consume the return on their saving and die. The assets held by the old correspond to the capital stock in the economy.

3.2.2 Firms

The firm sector is identical to the one in the representative agent model and conditions (3.1)–(3.3) apply. Profits are distributed to old households.

3.2.3 Households

The dynamic budget constraints of a worker and a retiree at date t as well as the Euler equation of a young household are given by

$$k_{t+1} = w_t - c_{1,t}, \quad (3.12)$$

$$c_{2,t} = k_t(1 + r_t - \delta) + \pi_t, \quad (3.13)$$

$$u'(c_{1,t}) = \beta(1 + r_{t+1} - \delta)u'(c_{2,t+1}), \quad (3.14)$$

respectively. Here, $c_{1,t}$ and $c_{2,t}$ denote consumption at date t of a young and old household, respectively.

3.2.4 Market Clearing

Labor and capital market clearing requires that firms demand one unit of labor and k_t units of capital, implying the equilibrium conditions (3.6) and (3.7). By Walras' Law,

market clearing in all but one market implies that the remaining market clears as well if all agents satisfy their budget constraints. Combining (3.3), (3.6), (3.7), (3.12), and (3.13) and letting $c_t \equiv c_{1,t} + c_{2,t}$ yields

$$\begin{aligned} k_{t+1} &= w_t - c_{1,t} + k_t(1 + r_t - \delta) - c_{2,t} + \pi_t \\ &= k_t(1 + r_t - \delta) + w_t - c_t + f(k_t, 1) - k_t r_t - 1w_t. \end{aligned}$$

This simplifies to the same resource constraint as in the representative agent model,

$$k_{t+1} = k_t(1 - \delta) + f(k_t, 1) - c_t.$$

3.2.5 General Equilibrium

In general equilibrium, conditions (3.1)–(3.3), (3.6)–(3.7), and (3.12)–(3.14) (and thus, the resource constraint) hold simultaneously. These equilibrium conditions can be reduced to three core equations,

$$\begin{aligned} k_{t+1} &= k_t(1 - \delta) + f(k_t, 1) - c_{1,t} - c_{2,t}, \\ c_{2,t} &= k_t(1 + f_K(k_t, 1) - \delta), \\ u'(c_{1,t}) &= \beta(1 + f_K(k_{t+1}, 1) - \delta)u'(c_{2,t+1}), \end{aligned}$$

as well as five remaining conditions that determine r_t , w_t , π_t , K_t , and L_t . Compared with the representative agent model, an additional budget constraint is present; it determines how consumption is split between workers and retirees. The Euler equation characterizes the slope of the consumption profile over the household's life.

Conditional on k_t , the core equations pin down $c_{1,t}$, $c_{2,t}$, and k_{t+1} .² For an initial capital stock, k_0 , the core equations therefore completely pin down the equilibrium paths of capital and consumption over the infinite horizon.

An alternative representation of equilibrium uses the saving function. Let $a_{t+1} = a(w_t, R_{t+1})$ denote equilibrium saving of a worker. The saving function a combines the Euler equation and the intertemporal budget constraint which extends over two periods; it depends on lifetime wealth, given by the wage, and the interest rate. Combined with the equilibrium relations between factor prices and the capital-labor ratio, the saving function defines a law of motion for capital,

$$k_{t+1} = a(w_t, R_{t+1}) \text{ where } w_t = f_L(k_t, 1), R_{t+1} = 1 - \delta + f_K(k_{t+1}, 1). \quad (3.15)$$

Under certain functional form assumptions this law of motion can be solved in closed form.

Depending on preferences and technology the function $k_{t+1}(k_t)$ defined by (3.15) may intersect the 45 degree line never, once, or multiple times; accordingly, no steady state with a strictly positive capital stock, a unique such steady state, or multiple steady states may exist. A steady state is stable and non-oscillating if in a neighborhood

²The second equation pins down $c_{2,t}$. Since $c_{2,t+1} = k_{t+1}(1 + f_K(k_{t+1}, 1) - \delta)$ the first and third condition pin down $c_{1,t}$ and k_{t+1} .

around it, k_{t+1} increases in k_t , but by less than one-to-one. Writing (3.15) as $k_{t+1} = \tilde{a}(k_t, k_{t+1})$ the steady state is stable and non-oscillating if $0 \leq [\partial \tilde{a}(k_t, k_{t+1}) / \partial k_t] / (1 - [\partial \tilde{a}(k_t, k_{t+1}) / \partial k_{t+1}]) \leq 1$.

3.2.6 Analysis

In contrast to the representative agent model, households in the overlapping generations model are heterogeneous. As a consequence, average saving in the economy differs from the saving of young or old households, and the slope of the consumption profile of a young household need not match the slope of the aggregate consumption profile.

This has important implications for the steady state. While the first steady-state condition of the representative agent model, condition (3.10), also applies in the overlapping generations model, the second one, condition (3.11), does not. In the representative agent model, this second condition follows from the requirement that aggregate and thus, individual consumption is constant over time. In the overlapping generations model, in contrast, constancy of aggregate consumption (or of young-age consumption or old-age consumption) does not imply that the consumption profile of an individual household is flat over the life cycle. The steady-state capital stock, the fixed point of (3.15), therefore need not satisfy the modified golden rule. Depending on preferences and the production function, it can be smaller or—unlike in the representative agent model—larger than the golden-rule capital stock.

Figure 3.3 illustrates the transition dynamics from a low initial capital stock, assuming either a high ($\alpha = 0.3$, top panel) or low ($\alpha = 0.2$, bottom panel) capital share. (We posit Cobb-Douglas technology and logarithmic preferences and let $\beta = 0.98^{25}$ and $\delta = 1 - (1 - 0.05)^{25}$, such that one period in the model corresponds to 25 years.) The concave black schedules depict the steady-state resource constraint; the red and blue dots represent $c_{1,t}$ and $c_{2,t}$ respectively, and the black dots represent aggregate consumption, c_t . The top panel of the figure illustrates the transition when the capital share is high. Both young-age consumption, old-age consumption, and aggregate consumption increase during the transition. Over each life cycle, however, consumption does not increase. The young cohort at the beginning of the transition faces a gross interest rate approximately equal to β^{-1} and accordingly chooses a flat consumption profile. Subsequent young cohorts are richer because they earn higher wages but they face lower interest rates and thus choose downward sloping consumption profiles. The economy converges to a steady state satisfying $k < k^{gr}$.

The lower panel illustrates the transition when the capital share (and the golden-rule capital stock) is lower. Again, all consumption components increase during the transition but over each life cycle, consumption decreases. Moreover, wages relative to interest rates are higher than in the previous case, generating a stronger motive for life-cycle saving. The economy now converges to a steady state satisfying $k > k^{gr}$.

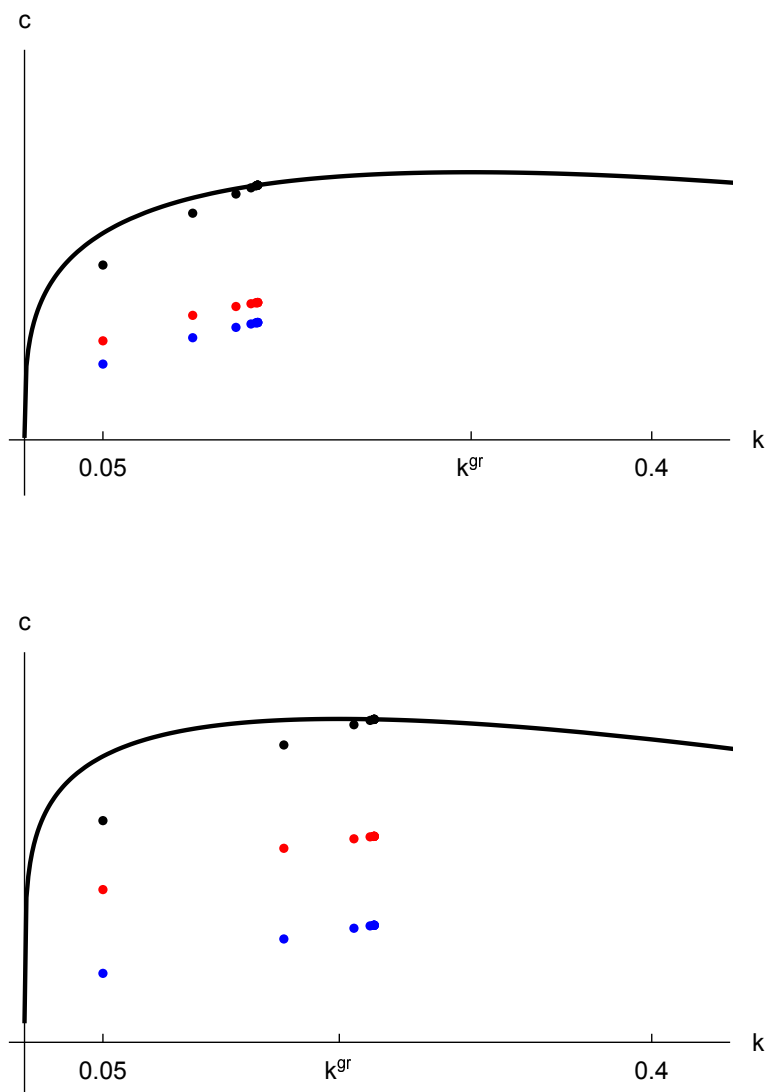


Figure 3.3: Transition dynamics in the OLG model: High (top) and low (bottom) capital share.

3.2.7 Pareto Optimality

To assess the efficiency of the steady state, note that an equilibrium cannot be Pareto optimal if the steady-state capital stock exceeds the golden-rule level such that $f_K(k, 1) < \delta$. Intuitively, with $k > k^{gr}$, net investment destroys resources available for consumption; a reduction of the capital stock makes everybody better off because it frees resources for consumption both immediately, due to the reduction of investment, and in all future periods, because the lower capital stock implies a stronger reduction of replacement investment than of output. With $dc/dk < 0$, the allocation is *dynamically inefficient*.

To take a stark example, suppose that capital does not contribute to production at all but depreciates at rate $\delta > 0$, that is saving amounts to storage and a fraction of the stored goods spoils. The production function then takes the form $f(K_t, L_t) = L_t$ and the dynamic inefficiency condition, $f_K(K_t, L_t) < \delta$, is met. Any positive level of storage amounts to a waste of resources, and aggregate consumption is maximized in the absence of storage, $dc/dk < 0$.

Since excessive capital accumulation amounts to a waste of resources a social planner avoids it. In the decentralized equilibrium, in contrast, young households may over accumulate capital because saving is the only way for them to smooth consumption over the life cycle. As a consequence, the market outcome in the overlapping generations economy need not be efficient, unlike in the representative agent model.

The assumptions underlying the first welfare theorem are not satisfied in a dynamically inefficient overlapping generations economy. If $f_K < \delta$ such that $R < 1$ then the market value of the endowment stream is infinite. The same holds true in an even simpler overlapping generations economy without capital where each cohort has an endowment in the first period of life and values the sum of life-time consumption.³ A competitive equilibrium in this economy is an allocation where each cohort consumes its own endowment and gross interest rates that equal unity. But a Pareto improvement could be achieved, for example, by letting each cohort consume the endowment of the subsequent cohort and in addition, letting the first cohort also consume its own endowment. Note that again, the market value of the endowment stream in the (dynamically inefficient) competitive equilibrium is infinite.

A transfer scheme akin to a pay-as-you-go financed *social security* system can cure dynamic inefficiency. Suppose that starting from date $t = 0$, young households pay a transfer b to the old. The old at date $t = 0$ clearly benefit from this arrangement since they receive a transfer without ever having contributed. But later cohorts benefit as well because the transfer scheme reduces the need for wasteful saving, freeing up resources for consumption.

This can most clearly be seen in the storage example with depreciation where saving a pays a return $1 - \delta < 1$. With transfers b , young-age consumption is given by $w - a - b$ and old-age consumption equals $a(1 - \delta) + b$. A marginal increase of trans-

³The argument is unchanged if all “cohorts” are infinitely lived but value consumption only in two successive periods. The inefficiency of equilibrium therefore is not the consequence of a lack of opportunities to trade.

fers improves welfare of current and future generations if $-u'(w - a - b) + \beta u'(a(1 - \delta) + b) \geq 0$ (using the envelope condition). From the Euler equation, $-u'(w - a - b) + \beta(1 - \delta)u'(a(1 - \delta) + b) = 0$ as long as households save. Comparing the two expressions, we conclude that the introduction of the transfer scheme improves welfare of all generations born at date $t = -1$ or later as long as the Euler equation holds, which is true as long as households save.

In general, transfers from young to old households lead to a Pareto improvement whenever the marginal product of capital in the initial equilibrium is lower than the depreciation rate, and when the economy has an infinite horizon. If the economy has a finite horizon, in contrast, then transfers do not lead to a Pareto improvement. Intuitively, if there exists a last period then transferring resources from the young to the old hurts the young in the last period. Formally, with a finite number of periods the market value of the endowment stream is finite as well, independently of market prices; the first theorem of welfare economics then implies efficiency of the decentralized equilibrium.

3.2.8 Population Growth

Suppose that the cohort size varies over time. Let $N_{1,t}$ and $N_{2,t}$ denote the mass of young and old households at date t , respectively, and let $\nu_t \equiv N_{1,t}/N_{2,t}$. Maintaining the definition of k_t as capital stock per worker as well as $c_{1,t}$ and $c_{2,t}$ as per-capita consumption, the budget constraints in equilibrium now read

$$\begin{aligned} c_{1,t} &= w_t - k_{t+1}\nu_{t+1}, \\ c_{2,t} &= k_t(1 + r_t - \delta)\nu_t \end{aligned}$$

and yield the resource constraint

$$\nu_{t+1}k_{t+1} = k_t(1 - \delta) + f(k_t, 1) - c_{1,t} - c_{2,t}/\nu_t.$$

The steady-state condition for dynamic inefficiency generalizes to

$$f_K(k, 1) < \delta + \nu - 1,$$

relating the net marginal product of capital to the net growth rate in the economy.

3.3 Bibliographic Notes

The representative agent, Ramsey, or neoclassical growth model is due to Ramsey (1928), Cass (1965), and Koopmans (1965). The overlapping generations model builds on Allais (1947), Samuelson (1958), and Diamond (1965). Modigliani and Brumberg (1954) discuss life cycle saving as well as the aggregation of heterogeneous consumption profiles. The analysis of dynamic inefficiency in the endowment economy is due to Shell (1971). Cass (1972) analyzes capital over accumulation.

Beyond the material covered in the chapter, Blanchard and Weil (1992) analyze dynamic efficiency in stochastic environments. Yaari (1965) and Blanchard (1985) analyze models of “perpetual youth” where households face a constant probability of death while new cohorts enter the economy.

Chapter 4

Risk

With risk, income and consumption streams may be random. This introduces new elements in the household's consumption-saving tradeoff. We study this tradeoff in two environments, with *incomplete markets* and *complete markets* respectively, depending on the number of assets with linearly independent returns. Thereafter, we analyze risk sharing and study how uninsurable income risk affects capital accumulation. Throughout, we assume that households evaluate random consumption streams according to the expected utility criterion.

4.1 Consumption, Saving, and Insurance

4.1.1 Incomplete Markets

Suppose that there are two periods. In the second period, one of two histories is realized, $\epsilon^1 = h$ or $\epsilon^1 = l$, with probability $\pi(h)$ and $\pi(l)$ respectively. (At date $t = 1$, the history and the state of nature are identical.) The wage in the second period, $w_1(\epsilon^1)$, depends on the history; it equals $w_1(h)$ or $w_1(l)$. The household has access to one asset, a_1 , and solves

$$\begin{aligned} \max_{a_1, c_0, c_1(h), c_1(l)} \quad & u(c_0) + \beta(\pi(h)u(c_1(h)) + \pi(l)u(c_1(l))) \\ \text{s.t.} \quad & a_1 = w_0 - c_0, \quad c_1(\epsilon^1) = a_1 R_1 + w_1(\epsilon^1). \end{aligned}$$

Note that the return on saving is not history-contingent, in contrast to the wage. This assumption is not important; what is crucial is that fewer assets than states of nature are available. Note also that consumption in the second period is history-contingent.

Substituting the second-period dynamic budget constraints into the first-period dynamic budget constraint, we find separate intertemporal budget constraints for each history,

$$c_0 + \frac{c_1(h)}{R_1} = w_0 + \frac{w_1(h)}{R_1} \quad \text{and} \quad c_0 + \frac{c_1(l)}{R_1} = w_0 + \frac{w_1(l)}{R_1}.$$

If the second-period wage assumes the high value, then the intertemporal budget constraint restricts the present value of lifetime consumption expenditures to not exceed

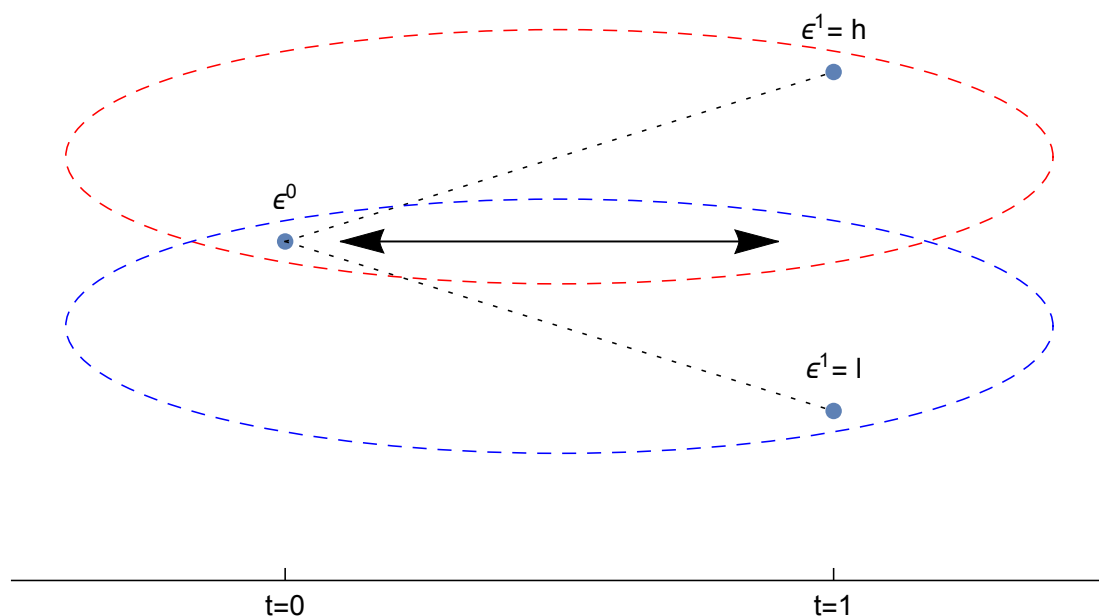


Figure 4.1: Incomplete markets: Two intertemporal budget constraints and one adjustment margin.

$w_0 + w_1(h)/R_1$. If, in contrast, it assumes the low value then the present value must not exceed $w_0 + w_1(l)/R_1$.

The household faces incomplete markets because it cannot exchange consumption in history h against consumption in history l . Figure 4.1 illustrates this. The dashed ellipses indicate the range of the two intertemporal budget constraints: One connects the initial period and the high state in the second period, the other the initial period and the low state. The arrows indicate the household's single margin of adjustment, corresponding to the choice of a_1 : Resources can be shifted over time—saving reduces c_0 and increases both $c_1(h)$ and $c_1(l)$ —but not to a specific node in the event tree.

Since the marginal benefit of saving accrues in more than one history, the *stochastic Euler equation* implied by the household's program involves an expectation operator,

$$u'(c_0) = \beta R_1 \mathbb{E}_0 \left[u'(c_1(\epsilon^1)) \right].$$

Intuitively, the cost of saving represented on the left-hand side is balanced with the average benefit across histories represented on the right-hand side.

Precautionary Saving

Assume that $\beta R_1 = 1$ such that the Euler equation reduces to $u'(c_0) = \mathbb{E}_0 [u'(c_1(\epsilon^1))]$. Without risk, the equilibrium consumption profile would be flat in this case. With risk, in contrast, it cannot be flat for all ϵ^1 because $w_1(\epsilon^1)$ is stochastic; in fact, the consumption profile generally is not even flat on average. To see this, assume that preferences are not only strictly concave, $u' > 0, u'' < 0$, as usual, but marginal utility

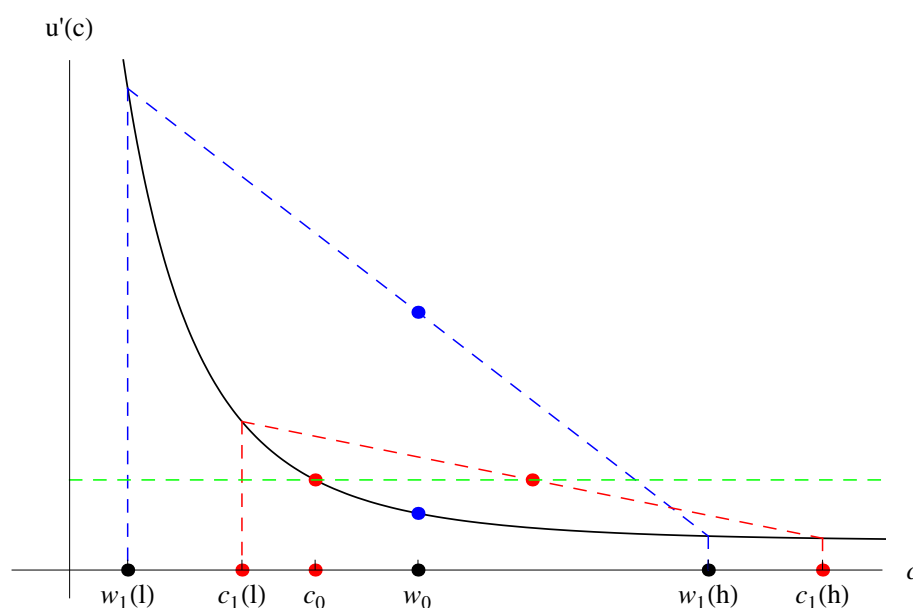


Figure 4.2: Convex marginal utility and income risk imply precautionary saving.

also is convex, $u''' > 0$. Most plausible period utility functions satisfy this condition. By Jensen's inequality we then have $\mathbb{E}_0 [u'(c_1(\epsilon^1))] > u'(\mathbb{E}_0 [c_1(\epsilon^1)])$ and the Euler equation therefore stipulates $u'(c_0) > u'(\mathbb{E}_0 [c_1(\epsilon^1)])$ or $c_0 < \mathbb{E}_0 [c_1(\epsilon^1)]$. We conclude that convex marginal utility implies strictly positive average consumption growth, in spite of $\beta R_1 = 1$; saving is higher than in the absence of risk, reflecting a *precautionary saving* motive or *prudence*. In contrast, linear marginal utility implies $c_0 = \mathbb{E}_0 [c_1(\epsilon^1)]$ and concave marginal utility implies $c_0 > \mathbb{E}_0 [c_1(\epsilon^1)]$.

Figure 4.2 illustrates the precautionary saving motive. The figure plots the marginal utility function (in black) against consumption. Suppose first that the wage is deterministic and equal to w_0 in both periods. Since $\beta R_1 = 1$ optimal consumption then equals w_0 in both periods and saving equals zero. Consider next the case of interest with a risky wage in the second period, indicated by black dots. If the household continues not to save then $c_0 = w_0$, $c_1(h) = w_1(h)$, and $c_1(l) = w_1(l)$. Due to the convexity of the marginal utility function, expected marginal utility of second-period consumption, indicated by the upper blue dot, exceeds marginal utility of first-period consumption, indicated by the lower blue dot, and the Euler equation is violated. Intuitively, the "downside" risk for consumption affects average marginal utility more strongly than the "upside" risk. To satisfy the Euler equation, saving must rise, first-period consumption must fall to c_0 , and history-contingent second-period consumption must rise to $c_0(l)$ or $c_0(h)$, all indicated by the lower red dots. In equilibrium, marginal utility in the first period and expected marginal utility in the second period, indicated by the upper red dots, coincide.

Note that the effect of risk on average marginal utility falls with household wealth. As a consequence, a richer household engages in less precautionary saving.

Certainty Equivalence

While strict convexity of the marginal utility function is plausible it renders solving the model difficult. Linear marginal utility, which is associated with a quadratic utility function, simplifies the analysis—at the cost of abstracting from precautionary saving. It implies $\mathbb{E}_0[u'(c_1(\epsilon^1))] = u'(\mathbb{E}_0[c_1(\epsilon^1)])$ and thus, that the Euler equation reduces to $c_0 = \mathbb{E}_0[c_1(\epsilon^1)] + \phi$ where ϕ equals zero when $\beta R_1 = 1$. This is an instance of *certainty equivalence* that is, the optimality conditions only depend on the expected value of the variable of interest (here, consumption).

To appreciate the gain in tractability due to certainty equivalence, consider a three-period setting with quadratic utility and $\beta R_t = 1$ at all times. The intertemporal budget constraint then reads

$$c_0 + \beta c_1(\epsilon^1) + \beta^2 c_2(\epsilon^2) = w_0 + \beta w_1(\epsilon^1) + \beta^2 w_2(\epsilon^2)$$

and the Euler equations reduce to $c_0 = \mathbb{E}_0[c_1(\epsilon^1)]$ and $c_1(\epsilon^1) = \mathbb{E}_1[c_2(\epsilon^2)]$. Equilibrium consumption thus follows a *random walk*. Using the law of iterated expectations, we can combine these results to find

$$c_0(1 + \beta + \beta^2) = w_0 + \beta \mathbb{E}_0[w_1(\epsilon^1)] + \beta^2 \mathbb{E}_0[w_2(\epsilon^2)].$$

At date $t = 1$, history ϵ^1 the intertemporal budget constraint conditional on saving in the initial period ($a_1 = w_0 - c_0$) reads

$$c_1(\epsilon^1) + \beta c_2(\epsilon^2) = (w_0 - c_0)\beta^{-1} + w_1(\epsilon^1) + \beta w_2(\epsilon^2) \quad \forall \epsilon^2 | \epsilon^1,$$

and the Euler equation is given by $c_1(\epsilon^1) = \mathbb{E}_1[c_2(\epsilon^2)]$. Taking expectations and combining the two conditions yields

$$c_1(\epsilon^1)(1 + \beta) = (w_0 - c_0)\beta^{-1} + w_1(\epsilon^1) + \beta \mathbb{E}_1[w_2(\epsilon^2)].$$

Comparing the results for c_0 conditional on information at date $t = 0$ and for $c_1(\epsilon^1)$ conditional on information at date $t = 1$, history ϵ^1 we note that

$$(c_1(\epsilon^1) - c_0)(1 + \beta) = (\mathbb{E}_1 - \mathbb{E}_0)[w_1(\epsilon^1) + \beta w_2(\epsilon^2)].$$

That is, the sign and magnitude of the innovation $c_1(\epsilon^1) - \mathbb{E}_0[c_1(\epsilon^1)]$ reflects how the expected present discounted value of income in and after date $t = 1$ changes as the information set changes from date $t = 0$ to date $t = 1$, history ϵ^1 .

Risk of Binding Borrowing Constraint

A binding borrowing constraint reduces consumption. It also affects consumption earlier in time, before the constraint binds. In a stochastic environment, this effect is present whenever a borrowing constraint may bind with strictly positive probability. We illustrate this in a three-period setting with stochastic income $w_1(\epsilon^1)$ in the second period and non-stochastic income w_0 and w_2 otherwise. For simplicity, we let

$\beta = 1$ and assume that gross returns also equal unity. Only borrowing at date $t = 1$ is prohibited, $a_2(\epsilon^1) \geq 0$.

We start by deriving the value function at date $t = 1$, history ϵ^1 , when uncertainty is resolved. In states where $a_1 + w_1(\epsilon^1) \geq w_2$ the preferred level of a_2 is positive and the borrowing constraint does not bind. Consumption in the second and third period is equal in this case and given by $(a_1 + w_1(\epsilon^1) + w_2)/2$. In states where $a_1 + w_1(\epsilon^1) < w_2$, in contrast, the borrowing constraint does bind and consumption in the second and third period equals $a_1 + w_1(\epsilon^1)$ and w_2 , respectively. The value function thus equals

$$V_1(a_1 + w_1(\epsilon^1)) = \begin{cases} u(a_1 + w_1(\epsilon^1)) + u(w_2) & \text{if } w_1(\epsilon^1) < w_2 - a_1 \\ 2 \cdot u\left(\frac{a_1 + w_1(\epsilon^1) + w_2}{2}\right) & \text{if } w_1(\epsilon^1) \geq w_2 - a_1 \end{cases}.$$

Note that the derivative of the value function has a kink at the critical value $a_1 + w_1(\epsilon^1) = w_2$ below which consumption smoothing is infeasible: $\lim_{\delta \downarrow 0} V_1''(w_2 - \delta) = u''(w_2)$ whereas $V_1''(w_2) = u''(w_2)/2$. That is, the derivative is convex around the critical level, independently of whether marginal utility is convex or not; all that is required for the convexity of V' is that preferences are strictly concave.

Consider now the effect of the potentially binding borrowing constraint at date $t = 1$ on saving in the initial period, a_1 . While the household's program

$$\max_{a_1} u(w_0 - a_1) + \mathbb{E}_0[V_1(a_1 + w_1(\epsilon^1))]$$

yields the usual Euler equation, $u'(c_0) = \mathbb{E}_0[V_1'(a_1 + w_1(\epsilon^1))]$, the convexity of V' leads the household to save more at date $t = 0$ than if no risk of a binding borrowing constraint were present. The intuition mirrors the one for precautionary saving although it is the risk of a binding borrowing constraint in combination with strictly concave preferences—not convexity of marginal utility—which drives the result.

Buffer Stock Saving

Consider an impatient household in an environment with constant interest rates, $\beta R < 1$. Absent risk, this household would choose a declining consumption path. With risk, in contrast, the precautionary saving motive or the risk of a future binding borrowing constraint work in the opposite direction and encourage saving.

The net effect on saving may depend on household wealth. If the marginal propensity to consume falls with household wealth then the two motives encouraging saving gain in strength as the household impoverishes: For given future income risk, less wealth translates into higher consumption risk and thus, a stronger saving motive. The wealth dependent saving motive on the one hand and impatience on the other give rise to a target ratio of financial assets to average income. During good times, the household builds up a *buffer stock* of financial assets from which it draws during bad times.

4.1.2 Complete Markets

Turning to complete markets, consider again the environment with two periods and two histories. In contrast to the incomplete market setting, the household now has access to two assets with linearly independent returns. For simplicity, we assume that these two assets are *Arrow-Debreu securities* that is, securities that only pay off in one history each (we relax this assumption later). We denote by a_1^1 the quantity of the first Arrow-Debreu security that pays off if and only if $\epsilon^1 = h$, and we denote the history-dependent return on this security by $R_1^1(\epsilon^1)$ with $R_1^1(l) = 0$. Similarly, a_1^2 denotes the quantity of the second Arrow-Debreu security that pays off if and only if $\epsilon^1 = l$, and its return is denoted $R_1^2(\epsilon^1)$ with $R_1^2(h) = 0$. The household's program reads

$$\begin{aligned} \max_{a_1^1, a_1^2, c_0, c_1(h), c_1(l)} \quad & u(c_0) + \beta(\pi(h)u(c_1(h)) + \pi(l)u(c_1(l))) \\ \text{s.t.} \quad & a_1^1 + a_1^2 = w_0 - c_0, \quad c_1(\epsilon^1) = a_1^1 R_1^1(\epsilon^1) + a_1^2 R_1^2(\epsilon^1) + w_1(\epsilon^1). \end{aligned}$$

As in the incomplete-market setting, three dynamic budget constraints bind. In contrast to the incomplete-market setting, however, these three constraints can be combined into a single intertemporal budget constraint rather than separate ones for each history:

$$c_0 + \frac{c_1(h)}{R_1^1(h)} + \frac{c_1(l)}{R_1^2(l)} = w_0 + \frac{w_1(h)}{R_1^1(h)} + \frac{w_1(l)}{R_1^2(l)}.$$

The situation is akin to a static environment where the household can exchange all goods ($c_0, c_1(h)$, and $c_1(l)$) against each other—the household faces complete markets. In particular, and in contrast to the incomplete-market setting, the two assets do not only allow the household to shift resources across time (that is, exchange c_0 against a *bundle* of $c_1(h)$ and $c_1(l)$) but also to specific nodes in the event tree. Equivalently, they allow to shift resources across histories at date $t = 1$, by buying less of one Arrow-Debreu security and more of the other. Since consumption in the two histories can be chosen independently of each other the household may achieve full insurance ($c_1(h) = c_1(l)$) although $w_1(h) \neq w_1(l)$, unlike in the incomplete-market case.

Figure 4.3 illustrates the complete-market setting. The dashed ellipse indicates the range of the single intertemporal budget constraint that connects the initial period and both states in the second period. The arrows indicate the two margins of adjustment, corresponding to the choices of a_1^1 and a_1^2 .

The first-order conditions of the program with Arrow-Debreu securities are given by the Euler equations

$$u'(c_0) = \beta R_1^1(h)\pi(h)u'(c_1(h)) \quad \text{and} \quad u'(c_0) = \beta R_1^2(l)\pi(l)u'(c_1(l)).$$

These conditions do not contain an expectation operator because the choice of a_1^1 or a_1^2 affects second-period consumption only in one history each. If returns are actuarially fair that is, return differentials compensate for risk such that $R_1^1(h)/R_1^2(l) = \pi(l)/\pi(h)$, then it is optimal to smooth consumption across states, $c_1(h) = c_1(l)$. If, moreover, $\beta R_1^1(h)\pi(h) = 1$, then consumption is perfectly smoothed over time as well, unlike in the incomplete-market economy.

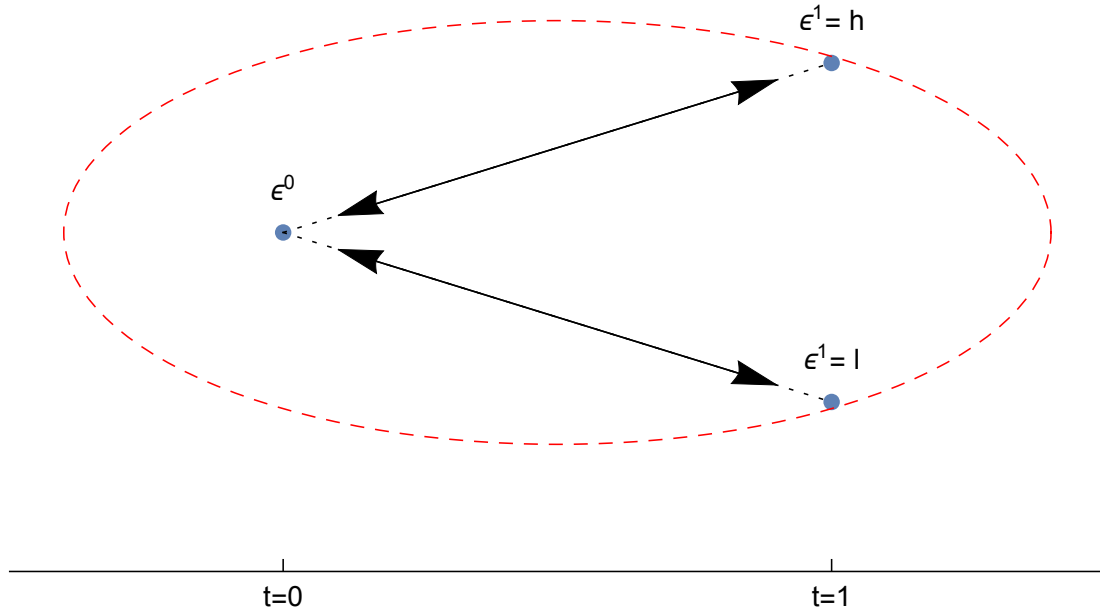


Figure 4.3: Complete markets: One intertemporal budget constraint and two adjustment margins.

Generalizations

Market completeness does not require the existence of Arrow-Debreu securities. It only requires as many assets with linearly independent returns as states of nature (which is guaranteed with a complete set of Arrow-Debreu securities). To understand the independence requirement, consider a general return structure with $R_1^i(\epsilon^1) \geq 0, i = 1, 2; \epsilon^1 = h, l$. (With Arrow-Debreu securities, $R_1^1(l) = R_1^2(h) = 0$.) The dynamic budget constraints in the second period can be expressed as

$$\begin{bmatrix} c_1(h) - w_1(h) \\ c_1(l) - w_1(l) \end{bmatrix} = \begin{bmatrix} R_1^1(h) & R_1^2(h) \\ R_1^1(l) & R_1^2(l) \end{bmatrix} \begin{bmatrix} a_1^1 \\ a_1^2 \end{bmatrix}.$$

If the return vectors $R_1^1(\epsilon^1)$ and $R_1^2(\epsilon^1)$ are linearly independent then the matrix on the right-hand side has full rank and its determinant, $D = R_1^1(h)R_1^2(l) - R_1^1(l)R_1^2(h)$, differs from zero. The equation thus can be solved for a_1^1 and a_1^2 . Substituting the resulting expressions into the first-period dynamic budget constraint yields the single intertemporal budget constraint

$$w_0 - c_0 + (w_1(h) - c_1(h)) \frac{R_1^2(l) - R_1^1(l)}{D} + (w_1(l) - c_1(l)) \frac{R_1^1(h) - R_1^2(h)}{D} = 0.$$

(When $R_1^1(l) = R_1^2(h) = 0$, this reduces to the constraint in the case with Arrow-Debreu securities.)

The term $(R_1^2(l) - R_1^1(l))/D$ in the intertemporal budget constraint represents the price of second-period consumption in history h , expressed in terms of first-period consumption. To see this, note that purchasing ϕ units of the first asset and $-\phi R_1^1(l)/R_1^2(l)$

units of the second yields a return of $\phi(R_1^1(h) - R_1^2(h)R_1^1(l)/R_1^2(l))$ in history h and zero in history l . To secure one additional unit of consumption in history h , the household thus must acquire $\phi = (R_1^1(h) - R_1^2(h)R_1^1(l)/R_1^2(l))^{-1} = R_1^2(l)/D$ units of the first asset and $-R_1^1(l)/D$ units of the second, at a cost of $(R_1^2(l) - R_1^1(l))/D$.¹ Similarly, $(R_1^1(h) - R_1^2(h))/D$ represents the price of consumption in history l .

In an interior equilibrium, the Euler equations now read

$$\begin{aligned} u'(c_0) &= \beta(R_1^1(h)\pi(h)u'(c_1(h)) + R_1^1(l)\pi(l)u'(c_1(l))), \\ u'(c_0) &= \beta(R_1^2(h)\pi(h)u'(c_1(h)) + R_1^2(l)\pi(l)u'(c_1(l))). \end{aligned}$$

Linear combinations of these equations yield Euler equations akin to those in the environment with Arrow-Debreu securities. For example, multiplying the first equation by $R_1^2(l)$ and the second by $-R_1^1(l)$ and summing yields

$$u'(c_0) = \beta \frac{D}{R_1^2(l) - R_1^1(l)} \pi(h)u'(c_1(h)).$$

In a multi-period environment with many goods, market completeness does not require that all history-contingent goods in all periods can be traded in the initial period (either by means of Arrow-Debreu securities or combinations of assets with linearly independent returns). What is needed instead is that at least one good can be traded contingently in each period and history, and that complete spot markets for all goods are open in all periods and histories.

4.1.3 General Case

Consider a two-period setup with a finite number of histories and a finite number of assets indexed by i . Markets may be complete or incomplete. The household's program reads

$$\begin{aligned} \max_{c_0, \{a_1^i\}_i, \{c_1(\epsilon^1)\}_{\epsilon^1}} \quad & u(c_0) + \beta \mathbb{E}_0 [u(c_1(\epsilon^1))] \\ \text{s.t.} \quad & \sum_i a_1^i = w_0 - c_0, \quad c_1(\epsilon^1) = \sum_i a_1^i R_1^i(\epsilon^1) + w_1(\epsilon^1). \end{aligned}$$

For each asset i that the household purchases or sells the corresponding Euler equation

$$u'(c_0) = \beta \mathbb{E}_0 [u'(c_1(\epsilon^1)) R_1^i(\epsilon^1)]$$

¹When $R_1^1(l) = R_1^2(l)$ but $D \neq 0$ then the return on one asset strictly dominates the return on the other: In history l both vehicles generate the same return, but in history h one generates a strictly higher return than the other (if $D > 0$ the first asset returns more, if $D < 0$ the second does). Buying the asset with the strictly higher return and selling the vehicle with the lower return allows to increase consumption in history h without having to give up consumption in the initial period or in history l ; the price of consumption in state h therefore equals zero.

holds. Expressed differently, $1 = \mathbb{E}_0[m_1(\epsilon^1)R_1^i(\epsilon^1)]$ where $m_1(\epsilon^1) \equiv \beta u'(c_1(\epsilon^1))/u'(c_0)$ denotes the household's marginal rate of substitution. Note that $\sum_i a_1^i = \mathbb{E}_0[m_1(\epsilon^1) \sum_i a_1^i R_1^i(\epsilon^1)]$ because the household either is invested in the asset i in which case $1 = \mathbb{E}_0[m_1(\epsilon^1)R_1^i(\epsilon^1)]$, or it is not invested in which case $a_1^i = 0$.

Multiplying the dynamic budget constraints at date $t = 1$ by $m_1(\epsilon^1)$ and taking expectations yields

$$\mathbb{E}_0[m_1(\epsilon^1)c_1(\epsilon^1)] = \mathbb{E}_0 \left[m_1(\epsilon^1) \sum_i a_1^i R_1^i(\epsilon^1) \right] + \mathbb{E}_0[m_1(\epsilon^1)w_1(\epsilon^1)].$$

Adding the dynamic budget constraint at date $t = 0$, we arrive at the equilibrium condition

$$c_0 + \mathbb{E}_0[m_1(\epsilon^1)c_1(\epsilon^1)] = w_0 + \mathbb{E}_0[m_1(\epsilon^1)w_1(\epsilon^1)]. \quad (4.1)$$

Condition (4.1) holds independently of whether markets are complete or incomplete. When markets are complete, (4.1) incorporates all equilibrium restrictions imposed by the intertemporal budget constraint and the Euler equations (except possibly a restriction that rates of return are given exogenously). When markets are incomplete, in contrast, condition (4.1) represents these equilibrium conditions only partially because it is an average of the multiple intertemporal budget constraints which bind individually.

To see this in the two special cases considered earlier, recall that in the saving problem with Arrow-Debreu securities the intertemporal budget constraint and Euler equations are given by

$$w_0 - c_0 + \frac{w_1(h) - c_1(h)}{R_1^1(h)} + \frac{w_1(l) - c_1(l)}{R_1^2(l)} = 0, \quad \frac{1}{R_1^1(h)} = \pi(h)m_1(h), \quad \frac{1}{R_1^2(l)} = \pi(l)m_1(l).$$

Substituting the latter into the former yields (4.1). In the saving problem with incomplete markets and a safe return, in contrast, the intertemporal budget constraints and Euler equation are given by

$$w_0 - c_0 + \frac{w_1(\epsilon^1) - c_1(\epsilon^1)}{R_1} = 0, \quad \frac{1}{R_1} = \mathbb{E}_0 \left[m_1(\epsilon^1) \right].$$

Substituting the latter into the former does not yield (4.1). But summing the two intertemporal budget constraints, weighted by the respective probabilities and marginal rates of substitution, does yield condition (4.1) once the Euler equation is imposed.

4.2 Risk Sharing

4.2.1 Borch Rule

Consider an economy with a large number of households that may buy and sell assets with contingent returns. In equilibrium at date t ,

$$1 = \mathbb{E}_t[m_{t+1}^h(\epsilon^{t+1})R_{t+1}^i(\epsilon^{t+1})]$$

for all assets i and all household types h where m_{t+1}^h denotes the marginal rate of substitution of type h . For an arbitrary type, l say, it follows that

$$\mathbb{E}_t[(m_{t+1}^l(\epsilon^{t+1}) - m_{t+1}^h(\epsilon^{t+1}))R_{t+1}^i(\epsilon^{t+1})] = 0$$

for all $h \neq l$.

When markets are complete the condition simplifies. To see this most directly, assume without loss of generality that the assets include a complete set of Arrow-Debreu securities. (Recall that the return of an Arrow-Debreu security equals zero in all histories except a single one.) The condition then reduces to $m_{t+1}^l(\epsilon^{t+1}) = m_{t+1}^h(\epsilon^{t+1})$ for all histories ϵ^{t+1} . If the two types share the same period utility function but not necessarily the time discount factor, this can be expressed as

$$\frac{u'(c_t^h(\epsilon^t))}{u'(c_t^l(\epsilon^t))} = \frac{\beta^h u'(c_{t+1}^h(\epsilon^{t+1}))}{\beta^l u'(c_{t+1}^l(\epsilon^{t+1}))}. \quad (4.2)$$

Condition (4.2), which is referred to as *Borch rule*, states that households *share risk*—whenever marginal utility of one household is high the same holds true for the other. The ratio of marginal utilities reflects differences in wealth and patience. When households have identical discount factors and homothetic period utility functions then their consumption ratios are constant over time and across histories.

Risk sharing is Pareto optimal. To see this, consider the problem of maximizing the welfare of type l subject to the other types attaining given levels of welfare, $\{\bar{U}^h\}$. The Lagrangian reads

$$\begin{aligned} \mathcal{L} = & \mathbb{E}_0 \left[\sum_{t=0}^{\infty} (\beta^l)^t u(c_t^l(\epsilon^t)) \right] + \sum_{h \neq l} \lambda^h \mathbb{E}_0 \left[\sum_{t=0}^{\infty} (\beta^h)^t u(c_t^h(\epsilon^t)) - \bar{U}^h \right] \\ & + \sum_{t, \epsilon^t} \mu_t(\epsilon^t) \left\{ \dots - c_t^l(\epsilon^t) - \sum_{h \neq l} c_t^h(\epsilon^t) + \dots \right\}, \end{aligned}$$

where we assume that the number of households is the same across types; λ^h denotes the multiplier associated with the reservation utility requirement for household type h ; and $\mu_t(\epsilon^t)$ denotes the multiplier associated with the resource constraint. We do not need to be specific about the production side of the economy, thus the dots. Differentiating yields

$$(\beta^l)^t u'(c_t^l(\epsilon^t)) = \lambda^h (\beta^h)^t u'(c_t^h(\epsilon^t))$$

which implies the risk sharing condition (4.2).

4.2.2 Aggregate and Idiosyncratic Risk

Suppose that households have endowments, $w_t^h(\epsilon^t) \equiv w_t(\epsilon^t) + l_t^h(\epsilon^t)$, with an *aggregate* and an *idiosyncratic* or household specific component. The former, $w_t(\epsilon^t)$, is the same for all types while the latter, $l_t^h(\epsilon^t)$, varies across types. We assume that the sum

of the idiosyncratic components across all types equals zero in any history ϵ^t . Markets are complete and in equilibrium, households therefore share risk.

Let φ^h denote wealth of type h relative to average wealth. With identical and homothetic preferences condition (4.2) then implies $c_t^h(\epsilon^t) = \varphi^h c_t(\epsilon^t)$ where $c_t(\epsilon^t)$ denotes average consumption. Using the resource constraint, this yields

$$c_t^h(\epsilon^t) = \varphi^h w_t(\epsilon^t).$$

Note that consumption of all households is proportional to the average endowment in the economy. In other words, with complete markets consumption of households only reflects aggregate shocks and idiosyncratic income risk is fully diversified.

4.3 Uninsurable Income Risk And Capital Accumulation

In stark contrast to the environment with risk sharing we now consider a setting where insurance is ruled out. We assume that there is only idiosyncratic risk and analyze the consequences of missing insurance markets for capital accumulation.

4.3.1 Economy

The structure of the economy is the same as in the representative agent model of section 3.1, except for one difference: The time endowment of each household is random. Formally, there is a continuum of measure one of infinitely lived households, indexed by $h \in [0, 1]$. The time endowment of household h at date t is given by $1 + l_t^h$. It is strictly positive, bounded, and i.i.d. across households with minimum value $1 + \underline{l}$ and mean $\mathbb{E}_t[1 + l_{t+1}^h] = 1$. As a consequence, aggregate labor supply equals unity at all times. We assume that the time endowment follows a *Markov process* that is, the probability distribution of the endowment in a period only depends on its realized value in the preceding period.

Households have access to a risk-free asset with gross interest rate R_t . Net financial assets of households correspond to the economy's capital stock, k_t . Firms rent labor at the competitive wage w_t per unit of time, and capital at the competitive rate r_t . Capital depreciates at rate δ and thus, $R_t = 1 + r_t - \delta$.

We consider a *stationary equilibrium*: While the time endowment and assets of an individual household change from period to period, the joint distribution of time endowments and assets across the population is time invariant. Accordingly, the aggregate capital stock and thus, interest rates and wages are time invariant as well.

4.3.2 Households

Household h maximizes $\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t^h(\epsilon^t))$. Its dynamic budget constraint is given by

$$a_{t+1}^h(\epsilon^t) = a_t^h(\epsilon^{t-1})R + w(1 + l_t^h(\epsilon^t)) - c_t^h(\epsilon^t).$$

Consumption must be non-negative. As debt must be serviced under all circumstances this implies a *natural borrowing constraint* according to which debt cannot exceed the market value of future labor income in the worst possible history,

$$a_{t+1}^i(\epsilon^t) \geq \underline{a} \equiv -w(1+i)\frac{1}{R-1}.$$

In addition, a tighter borrowing constraint may bind. For example, if households are excluded from borrowing at all then the natural borrowing constraint is replaced by the restriction $a_{t+1}^h(\epsilon^t) \geq 0$.

At date t , history ϵ^t the state variables in the household's program are $(a_t^h(\epsilon^{t-1}), l_t^h(\epsilon^t))$ as well as the constant wage and interest rate, and the control variables are $(a_{t+1}^h(\epsilon^t), c_t^h(\epsilon^t))$. Let (a_\circ^h, l_\circ^h) denote household assets and the time endowment in the current period, and let (a_+^h, l_+^h) denote those objects in the subsequent period. Since the program is time autonomous the Bellman equation for household h reads

$$V(a_\circ^h, l_\circ^h; w, R) = \max_{a_+^h} u(a_\circ^h R + w(1+l_\circ^h) - a_+^h) + \beta \mathbb{E} \left[V(a_+^h, l_+^h; w, R) | l_\circ^h \right]$$

subject to the borrowing constraint; the expectation is conditional on l_\circ^h because the current time endowment may contain information about the probability distribution of next period's endowment (the Markov assumption implies that l_\circ^h contains all such information).

If ι risk were absent (or equivalently, if markets were complete and households insured each other), the household would face a risk-free, constant labor income stream. With $\beta R = 1$, its optimal consumption would be time invariant and equal to $a_\circ^h(R-1) + w$. With risk, in contrast, optimal consumption cannot be constant and finite. If it were, its value would have to be consistent with the worst case scenario of minimum time endowments forever after. But after a more favorable time endowment realization the household could increase consumption and this implies a contradiction.

When $\beta R \geq 1$, the household's first-order and envelope conditions yield

$$V_a(a_\circ^h, l_\circ^h; w, R) = u'(c_\circ^h) \geq \mathbb{E} \left[V_a(a_+^h, l_+^h; w, R) | l_\circ^h \right] = \mathbb{E} \left[u'(c_+^h) | l_\circ^h \right].$$

With $u' > 0$ and $u'' < 0$, optimal consumption stochastically converges to infinity in this case—it is not stationary—as the household accumulates more and more assets to *self insure* against low future time endowment realizations. Mathematically, the non-stationarity result follows from three observations. First, the Euler equation $u'(c_\circ^h) \geq \mathbb{E} [u'(c_+^h) | l_\circ^h]$ implies that marginal utility follows a submartingale, which converges. Second, marginal utility only converges if consumption converges, due to $u'' < 0$. Third, with finite asset holdings an income shock translates into a change of consumption.

We conclude that a stationary equilibrium requires $\beta R < 1$.

4.3.3 General Equilibrium

The stationary equilibrium in the hypothetical economy without risk (or with insurance) would satisfy $R = \beta^{-1} = 1 + r - \delta$, $f_K(k, 1) = r$, and $f_L(k, 1) = w$. Aggregate consumption would equal $k(R - 1) + w$.

In the stationary equilibrium in the economy with risk, in contrast, R must be strictly smaller than β^{-1} to clear the market for capital; otherwise the supply of capital would grow without bound while firms' demand would be bounded. Households accumulate assets when their time endowment is high and run them down when it is low. The capital stock in the economy is constant and since $R < \beta^{-1}$, it is strictly larger than in the economy without risk, and so are wages. Although the risk is purely idiosyncratic and washes out in the aggregate, self insurance gives rise to a higher capital stock.

Suppose for simplicity that the time endowment can assume m possible values and the asset holdings of a household n values; both m and n are finite. The $m \times m$ transition matrix Π^l whose rows sum to unity contains the transition probabilities of the time endowment; the (i, j) element of Π^l gives the probability that ι_+ takes the j -th of the m possible values conditional on ι taking the i -th such value.

The state (a^h, ι^h) of household h then can take mn values. Together with the transition matrix Π^l , the decision rules of households define a transition matrix for this state, Π say which is of size $mn \times mn$. Let Π^\top denote the transpose of Π , and let d of size $mn \times 1$ denote the probability distribution of households over the possible states; the elements of d sum to one. Note that conditional on d_\circ , the probability distribution in the subsequent period is given by $d_+ = \Pi^\top d_\circ$. A *stationary distribution* therefore satisfies $d = \Pi^\top d$; it is the normalized eigenvector associated with the unit eigenvalue of Π^\top .

4.4 Bibliographic Notes

Modigliani and Brumberg (1954) discuss the "precautionary motive" for saving and Friedman (1957, p. 16) discusses saving as a "reserve for emergencies." Leland (1968) relates the precautionary saving motive to the convexity of marginal utility, and Sandmo (1970) analyzes the differences between return and labor income risk. The analysis of the saving problem with quadratic utility is due to Hall (1978). Kimball (1990) defines prudence as the sensitivity of an optimal choice (here saving) to risk. Zeldes (1989*b*) simulates optimal consumption choices in the presence of a precautionary motive and finds that the marginal propensity to consume transitory income varies with the level of household assets. Zeldes (1989*a*) emphasizes that the risk of a binding borrowing constraint affects equilibrium consumption. Deaton (1991) analyzes the role of assets as "buffer stock" in a model with borrowing constraints, precautionary motive, and impatience. Carroll (1997) analyzes buffer stock saving in a model with precautionary motive and impatience and shows that households target a wealth-to-permanent-income ratio. Radner (1982) discusses equilibrium with sequential trading. Aiyagari (1994) analyzes idiosyncratic risk and capital accumulation in stationary equilibrium,

building on Bewley (1977; 1980; 1986) and Huggett (1993). Chamberlain and Wilson (2000) prove that consumption grows without bound if $\beta R \geq 1$.

Chapter 5

Asset Returns and Asset Prices

If a household invests in multiple assets it is indifferent between the investments at the margin. Assets with unequal payoff characteristics therefore trade at different prices. We derive the implications of household optimization and market clearing for asset returns and prices. Throughout we assume that households maximize expected utility. In appendix B.4 we discuss a model that relaxes this assumption.

5.1 Asset Pricing Kernel

Consider the equilibrium in an economy with two periods and risk. Markets may be complete or incomplete. We saw earlier (see subsection 4.1.3) that for each asset i and each household h that purchases or sells the asset, an Euler equation

$$u'(c_0^h) = \beta \mathbb{E}_0[u'(c_1^h(\epsilon^1))R_1^i(\epsilon^1)] \quad \text{or} \quad 1 = \mathbb{E}_0[m_1^h(\epsilon^1)R_1^i(\epsilon^1)]$$

holds where $m_1^h(\epsilon^1) \equiv \beta u'(c_1^h(\epsilon^1)) / u'(c_0^h)$ denotes household h 's marginal rate of substitution.

This has two important implications. First, when asset i is held by different households, l and n say, then the marginal rates of substitution of these households satisfy $\mathbb{E}_0[m_1^l(\epsilon^1)R_1^i(\epsilon^1)] = \mathbb{E}_0[m_1^n(\epsilon^1)R_1^i(\epsilon^1)]$ that is, the return weighted average marginal rates of substitution coincide. When l and n hold multiple assets then this imposes multiple cross-household restrictions on their marginal rates of substitution. When markets are complete, then the cross-household restrictions imply that in each history, both households have the same marginal rate of substitution, $m_1^l(\epsilon^1) = m_1^n(\epsilon^1)$ for all ϵ^1 . That is, the equilibrium marginal rates of substitution, $\{m_1(\epsilon^1)\}_{\epsilon^1}$, also referred to as the *asset pricing kernel* or *stochastic discount factor*, are unique in this case.

The second implication concerns average return differentials across assets, or excess returns to which we turn next.

5.2 Excess Returns

5.2.1 C-CAPM

When a household with marginal rates of substitution, $\{m_1(\epsilon^1)\}_{\epsilon^1}$, purchases or sells multiple assets, j and k say, then the returns on these assets satisfy $\mathbb{E}_0[m_1(\epsilon^1)R_1^j(\epsilon^1)] = \mathbb{E}_0[m_1(\epsilon^1)R_1^k(\epsilon^1)]$ that is, the kernel weighted average returns on the assets coincide. Expressing the equality as $\mathbb{E}_0[m_1(\epsilon^1)(R_1^j(\epsilon^1) - R_1^k(\epsilon^1))] = 0$ and using the definition of covariance yields

$$\mathbb{E}_0[m_1(\epsilon^1)]\mathbb{E}_0[R_1^j(\epsilon^1) - R_1^k(\epsilon^1)] + \text{Cov}_0[m_1(\epsilon^1), R_1^j(\epsilon^1) - R_1^k(\epsilon^1)] = 0.$$

Equilibrium therefore implies that the *expected returns* and *return covariances* of assets satisfy certain restrictions. The covariances measure how strongly asset returns covary with the marginal rate of substitution.

Suppose that asset f is risk-free, $R_1^f(\epsilon^1) = R_1^f$, and the household holds the risk-free asset such that $1 = \mathbb{E}_0[m_1(\epsilon^1)]R_1^f$. The *excess return* of asset i that is, the expected return net of the risk-free return thus satisfies

$$\mathbb{E}_0[R_1^i(\epsilon^1)] - R_1^f = -\frac{\text{Cov}_0[m_1(\epsilon^1), R_1^i(\epsilon^1)]}{\mathbb{E}_0[m_1(\epsilon^1)]} = -\text{Cov}_0[m_1(\epsilon^1), R_1^i(\epsilon^1)]R_1^f.$$

According to this *consumption capital asset pricing model* (C-CAPM) result, the excess return is proportional to the covariance between the asset's return and the marginal rate of substitution. Note that the excess return only compensates for correlation between the asset return and marginal utility, not for return risk per se.¹

Since β and $u'(c_0)$ in $m_1(\epsilon^1)$ are constants, the excess return effectively depends on the covariance between $R_1^i(\epsilon^1)$ and $u'(c_1(\epsilon^1))$. The asset pays zero excess return if this covariance is zero, for example because $u'(c_1(\epsilon^1))$ or $R_1^i(\epsilon^1)$ is deterministic. If the asset return covaries negatively with the marginal rate of substitution and thus (if utility is strictly concave) positively with $c_1(\epsilon^1)$, then the excess return is positive. Intuitively, the asset is a bad hedge in this case; it tends to pay more when the marginal benefit from additional resources is small. To induce the household to nevertheless hold the asset its return must be high. If the asset return covaries negatively with $c_1(\epsilon^1)$, in contrast, then the excess return is negative; when the asset is a good hedge it need not pay a high average return.

The C-CAPM implies a *mean-variance frontier* that gives a bound on the absolute value of an asset's excess return given the standard deviation of its return:

$$\begin{aligned} \mathbb{E}_0[R_1^i(\epsilon^1)] - R_1^f &= -\frac{\text{Cov}_0[m_1(\epsilon^1), R_1^i(\epsilon^1)]}{\mathbb{E}_0[m_1(\epsilon^1)]} \\ \Rightarrow |\mathbb{E}_0[R_1^i(\epsilon^1)] - R_1^f| &\leq \frac{\text{Std}_0[m_1(\epsilon^1)]\text{Std}_0[R_1^i(\epsilon^1)]}{\mathbb{E}_0[m_1(\epsilon^1)]}. \end{aligned}$$

¹Note also that $\text{Cov}_0[m_1(\epsilon^1), R_1^i(\epsilon^1)] / \text{Var}_0[m_1(\epsilon^1)]$ equals the projection of $R_1^i(\epsilon^1)$ on $m_1(\epsilon^1)$. The excess return is proportional to this projection.

Here, we use the fact that the covariance equals the product of the standard deviations and the correlation coefficient, and the latter lies between minus and plus one.

5.2.2 CAPM

The C-CAPM establishes a linear relation between the equilibrium excess return on an asset and the covariance between the asset return and the asset pricing kernel. The *capital asset pricing model* (CAPM), which precedes the C-CAPM, similarly establishes such a linear relation; but in the case of the CAPM it is the covariance between the asset return and the return on the *market portfolio* encompassing all risky assets—not the pricing kernel—which enters the relation.

Traditionally, the CAPM is derived by assuming that rather than consumption, a representative household values the mean return on its portfolio (positively) as well as the return variance (negatively). The first-order conditions with respect to the portfolio shares invested in risky assets then imply a linear relation between an asset's excess return and the covariance between the return and the household's risky portfolio return. Moreover, market clearing requires that the portfolio of risky assets held by the household corresponds to the market portfolio, and the CAPM follows.

Alternatively, the CAPM derives from the C-CAPM under the assumption that consumption is a linear function of the return on the market portfolio—the *market return* $R_1^m(\epsilon^1)$ —and the utility function is accurately approximated to the second order (or marginal utility to the first order). The asset pricing kernel $m_1(\epsilon^1)$ then is a linear function of $R_1^m(\epsilon^1)$ and letting ϕ denote a factor of proportionality, we have

$$\mathbb{E}_0[R_1^i(\epsilon^1)] - R_1^f = \text{Cov}_0 \left[R_1^m(\epsilon^1), R_1^i(\epsilon^1) \right] \phi R_1^f.$$

In particular, $\mathbb{E}_0[R_1^m(\epsilon^1)] - R_1^f = \text{Cov}_0[R_1^m(\epsilon^1), R_1^m(\epsilon^1)] \phi R_1^f$. It follows that

$$\mathbb{E}_0[R_1^i(\epsilon^1)] - R_1^f = \frac{\text{Cov}_0[R_1^m(\epsilon^1), R_1^i(\epsilon^1)]}{\text{Var}_0[R_1^m(\epsilon^1)]} (\mathbb{E}_0[R_1^m(\epsilon^1)] - R_1^f),$$

where the ratio on the right-hand side of the last equality represents asset i 's “beta,” the projection of $R_1^i(\epsilon^1)$ on $R_1^m(\epsilon^1)$. According to the CAPM, the excess return reflects the asset's beta and the market's excess return.

5.3 Asset Prices

To derive the implications of the C-CAPM for *asset prices*, we use the definition of a return: The gross rate of return between date t and $t + 1$, $R_{t+1}^i(\epsilon^{t+1})$, equals the payoff at date $t + 1$ relative to the asset price at date t , $p_t^i(\epsilon^t)$; and the payoff consists of the asset price, $p_{t+1}^i(\epsilon^{t+1})$, and the dividend, $d_{t+1}^i(\epsilon^{t+1})$:

$$R_{t+1}^i(\epsilon^{t+1}) \equiv \frac{p_{t+1}^i(\epsilon^{t+1}) + d_{t+1}^i(\epsilon^{t+1})}{p_t^i(\epsilon^t)}. \quad (5.1)$$

We can therefore rewrite the return condition, $1 = \mathbb{E}_t[m_{t+1}(\epsilon^{t+1})R_{t+1}^i(\epsilon^{t+1})]$, as

$$p_t^i(\epsilon^t) = \mathbb{E}_t[m_{t+1}(\epsilon^{t+1})(p_{t+1}^i(\epsilon^{t+1}) + d_{t+1}^i(\epsilon^{t+1}))].$$

Conditional on an asset pricing kernel and a probability distribution over histories, any asset with specified payoffs thus can be priced by computing the expectation of the kernel times the payoff.

While macroeconomists relate the asset pricing kernel to consumption, financial economists often take it as given or derive it from market prices and payoffs. Financial economists also use observed market prices and payoffs to price “new” securities such as derivatives. The *law of one price* states that portfolios with identical payoffs have the same price (unless households face portfolio restrictions). A *strong arbitrage* is a portfolio with a strictly negative price that pays off a non-negative amount in every history; it exists if the law of one price is violated. Equilibrium rules out the existence of a strong arbitrage (except in the presence of portfolio restrictions) and no-arbitrage conditions thus impose constraints on the prices of new securities. No-arbitrage conditions effectively impose the same structure as the assumption of equilibrium and strictly increasing utility functions.

5.3.1 Fundamental Value

With multiple periods, iterating the pricing equation forward yields

$$\begin{aligned} p_0^i &= \mathbb{E}_0 \left[m_1(\epsilon^1) \left(d_1^i(\epsilon^1) + \mathbb{E}_1 \left[m_2(\epsilon^2) \left(d_2^i(\epsilon^2) + \dots + \mathbb{E}_{T-1} \left[m_T(\epsilon^T) d_T^i(\epsilon^T) \right] \right) \right] \right) \right] \\ &\quad + \mathbb{E}_0 \left[m_1(\epsilon^1) \mathbb{E}_1 \left[m_2(\epsilon^2) \dots \mathbb{E}_{T-1} \left[m_T(\epsilon^T) p_T^i(\epsilon^T) \right] \right] \right] \\ &= \mathbb{E}_0 \left[\sum_{s=1}^T (m_1(\epsilon^1) \dots m_s(\epsilon^s)) d_s^i(\epsilon^s) \right] + \mathbb{E}_0 \left[(m_1(\epsilon^1) \dots m_T(\epsilon^T)) p_T^i(\epsilon^T) \right] \\ &= \mathbb{E}_0 \left[\sum_{s=1}^T \beta^s \frac{u'(c_s(\epsilon^s))}{u'(c_0)} d_s^i(\epsilon^s) \right] + \mathbb{E}_0 \left[\beta^T \frac{u'(c_T(\epsilon^T))}{u'(c_0)} p_T^i(\epsilon^T) \right], \end{aligned}$$

where we use the law of iterated expectations. The asset price has two components: The expected present discounted value of the dividend stream until date $t = T$, and the expected present discounted value of the price at this date. If T is the final period such that $p_T^i(\epsilon^T) = 0$ then the former component is the asset’s *fundamental value*.

If the asset has an infinite maturity then its price satisfies

$$p_0^i = \lim_{T \rightarrow \infty} \mathbb{E}_0 \left[\sum_{s=1}^T \beta^s \frac{u'(c_s(\epsilon^s))}{u'(c_0)} d_s^i(\epsilon^s) \right] + \lim_{T \rightarrow \infty} \mathbb{E}_0 \left[\beta^T \frac{u'(c_T(\epsilon^T))}{u'(c_0)} p_T^i(\epsilon^T) \right].$$

Again, it has two components, a fundamental value and a *bubble* component (the right-most term). Whether p_0^i exceeds the fundamental value depends on whether the bubble component is strictly positive and thus, whether $p_T^i(\epsilon^T)$ grows more quickly than $\beta^T u'(c_T(\epsilon^T)) / u'(c_0)$ shrinks as $T \rightarrow \infty$. We turn next to the question whether this is possible.

5.3.2 Bubbles

For simplicity, suppose that the utility function is linear and dividends are constant, $d_t^i(\epsilon^t) = d^i$, such that $m_t(\epsilon^t) = \beta$ and

$$p_0^i = \lim_{T \rightarrow \infty} \sum_{s=1}^T \beta^s d^i + \lim_{T \rightarrow \infty} \beta^T \mathbb{E}_0[p_T^i(\epsilon^T)].$$

One solution to this equation is a constant price equal to the fundamental value, $p_t^i = d^i \beta / (1 - \beta)$; the bubble component equals zero in this case. Another candidate solution is $p_t^i = d^i \beta / (1 - \beta) + \text{bubble}_t^i$ where $\{\text{bubble}_t^i\}_{t \geq 0}$ is a strictly positive sequence satisfying $\text{bubble}_t^i = \beta \text{bubble}_{t+1}^i$; that is, the bubble grows at the rate of interest. This candidate solution satisfies the asset pricing equation because

$$p_t^i = d^i \frac{\beta}{1 - \beta} + \text{bubble}_t^i = \beta d^i + \beta d^i \frac{\beta}{1 - \beta} + \beta \text{bubble}_{t+1}^i = \beta(d^i + p_{t+1}^i).$$

(Still other candidate solutions involve stochastic bubbles.)

To check whether the candidate solution with a bubble component is consistent with rational expectations, suppose first that the number of potential investors is finite. In this case it is impossible that all households purchasing the asset at a bubbly price expect somebody else to purchase it at an even higher bubbly price in the future. A bubbly price therefore is inconsistent with common knowledge in a rational expectations equilibrium.

Suppose next that new potential investors enter the economy as time progresses. A household may then purchase the asset at a bubbly price expecting to resell it to subsequent investors with similar expectations. When the interest rate strictly exceeds the economy's growth rate then such expectation formation cannot be rational; a bubble growing at the rate of interest would eventually outgrow the economy and newcomers would not be able to purchase the bubble any longer. But when the interest rate falls short of the growth rate, then a bubble may be sustained in rational expectations equilibrium.

Recall that the growth rate in a dynamically inefficient overlapping generations economy exceeds the interest rate. Such an environment therefore admits bubbles. In fact, a bubble can play exactly the same Pareto improving role as an inter generational transfer scheme (see p. 37): When old households sell a bubble to young ones the latter transfer resources to the former. This absorbs saving of the young cohort and reduces or eliminates capital over accumulation.

5.4 Term Structure of Interest Rates

The price of a risk-free one period bond that pays off unity in all histories, equals

$$p_t^{f1}(\epsilon^t) = \mathbb{E}_t \left[m_{t+1}(\epsilon^{t+1}) 1 \right].$$

The risk-free one period gross interest rate, $R_{t+1}^{f1}(\epsilon^t)$, equals the inverse of the bond price. This follows from the definition of a return, condition (5.1), or equivalently from the equilibrium condition

$$1 = \mathbb{E}_t \left[m_{t+1}(\epsilon^{t+1}) R_{t+1}^{f1}(\epsilon^t) \right] = \mathbb{E}_t \left[m_{t+1}(\epsilon^{t+1}) \right] R_{t+1}^{f1}(\epsilon^t).$$

Similarly, a risk-free s period bond that pays off unity in all histories ϵ^{t+s} is priced at

$$p_t^{fs}(\epsilon^t) = \mathbb{E}_t \left[m_{t+1}(\epsilon^{t+1}) \cdots m_{t+s}(\epsilon^{t+s}) 1 \right]$$

and the risk-free s period gross interest rate, $R_{t+s}^{fs}(\epsilon^t)$, equals the inverse of $p_t^{fs}(\epsilon^t)$.

Because $\{m_{t+1}(\epsilon^{t+1})\}_{\epsilon^{t+1}}$ affects both short- and longer-term interest rates these rates satisfy cross-restrictions. Consider $R_{t+1}^{f1}(\epsilon^t)$ and $R_{t+2}^{f2}(\epsilon^t)$:

$$\begin{aligned} \left(R_{t+2}^{f2}(\epsilon^t) \right)^{-1} &= \mathbb{E}_t \left[m_{t+1}(\epsilon^{t+1}) m_{t+2}(\epsilon^{t+2}) \right] = \mathbb{E}_t \left[m_{t+1}(\epsilon^{t+1}) \mathbb{E}_{t+1} \left[m_{t+2}(\epsilon^{t+2}) \right] \right] \\ &= \mathbb{E}_t \left[m_{t+1}(\epsilon^{t+1}) \left(R_{t+2}^{f1}(\epsilon^{t+1}) \right)^{-1} \right] \\ &= \left(R_{t+1}^{f1}(\epsilon^t) \right)^{-1} \mathbb{E}_t \left[\left(R_{t+2}^{f1}(\epsilon^{t+1}) \right)^{-1} \right] + \text{Cov}_t \left[m_{t+1}(\epsilon^{t+1}), \left(R_{t+2}^{f1}(\epsilon^{t+1}) \right)^{-1} \right]. \end{aligned}$$

Accordingly, there are two drivers of (the inverse of) the long-term interest rate, $R_{t+2}^{f2}(\epsilon^t)$. First, current and expected future (inverse) short-term rates. Second, a covariance term unless future short-term interest rates are uncorrelated with consumption. The *expectations hypothesis* abstracts from the second driver; it postulates that the *term premium* equals zero.

To compare the yields on bonds of different maturity it is useful to express interest rates over time intervals of the same length (e.g., on an annual basis). One may then represent the *term structure* of interest rates as the collection

$$\left\{ R_{t+1}^{f1}(\epsilon^t), \sqrt{R_{t+2}^{f2}(\epsilon^t)}, \sqrt[3]{R_{t+3}^{f3}(\epsilon^t)}, \dots \right\}.$$

The *yield curve* graphically represents the term structure; it plots annualized interest rates against time to maturity. According to the expectations hypothesis short-term interest rates are expected to rise in the future when $\sqrt[s]{R_{t+s}^{fs}(\epsilon^t)}$ increases in s that is, when the yield curve is upward sloping.

5.5 Asset Prices in an Endowment Economy

Every equilibrium model with a saving margin is an equilibrium model of asset prices. Once the equilibrium consumption processes are determined they imply asset pricing kernels which price arbitrary assets. In an economy with homogeneous households the

kernel is unique since all households have the same consumption process. In an endowment economy with homogeneous households the pricing is particularly straightforward because the equilibrium consumption process is exogenous, as we now show.

5.5.1 Economy

Consider an economy with a fixed capital stock that consists of a continuum of mass one of “trees.” Dividends—the “fruit” of the trees—are exogenous, stochastic, and cannot be stored. A continuum of mass one of infinitely lived homogeneous households owns the trees and consumes their fruit. Households have no other sources of income but may sell or purchase trees to/from each other. The budget constraint of household h reads

$$c_t^h(\epsilon^t) + p_t^{tr}(\epsilon^t) \left(tr_{t+1}^h(\epsilon^t) - tr_t^h(\epsilon^{t-1}) \right) = tr_t^h(\epsilon^{t-1}) d_t^{tr}(\epsilon^t),$$

where $c_t^h(\epsilon^t)$ denotes consumption; $p_t^{tr}(\epsilon^t)$ the tree price; $tr_{t+1}^h(\epsilon^t)$ the household's stock of trees between t and $t + 1$; and $d_t^{tr}(\epsilon^t)$ the dividend.

5.5.2 General Equilibrium

In equilibrium, each household owns the same number of trees, $tr_{t+1}^h(\epsilon^t) = 1$. Its desired asset holdings correspond to the tree multiplied by the tree's price, and its consumption corresponds to the fruit of a tree. Absent bubbles, the equilibrium price therefore satisfies

$$p_t^{tr}(\epsilon^t) = \mathbb{E}_t \left[\sum_{s=1}^{\infty} \beta^s \frac{u'(d_{t+s}^{tr}(\epsilon^{t+s}))}{u'(d_t^{tr}(\epsilon^t))} d_{t+s}^{tr}(\epsilon^{t+s}) \right].$$

While an individual household perceives its asset holdings and consumption to be endogenous, market clearing requires that each household owns one tree and thus, consumes the dividend in full. The market price given above induces this choice.

5.5.3 Pricing Other Assets

Using the equilibrium pricing kernel we can determine the price of arbitrary assets, including those that are in zero net supply and are not actually traded. For example, the price of a risk-free one period bond that pays off unity in all histories ϵ^{t+1} , is given by

$$p_t^{f1}(\epsilon^t) = \mathbb{E}_t \left[\beta \frac{u'(d_{t+1}^{tr}(\epsilon^{t+1}))}{u'(d_t^{tr}(\epsilon^t))} \right],$$

and the risk-free one period gross interest rate, $R_{t+1}^{f1}(\epsilon^t)$, equals the inverse of this price.

Suppose that dividends follow a Markov process with two states, d^h (high) or d^l (low), and symmetric transition probabilities of one half. In stationary equilibrium the price of trees and the risk-free one period interest rate are functions of the state that is,

the price equals $p^{tr,h}$ or $p^{tr,l}$ and the interest rate equals $R^{f1,h}$ or $R^{f1,l}$. With logarithmic preferences this implies the following equilibrium conditions:

$$\begin{aligned} p^{tr,h} &= d^h \frac{\beta}{1 - \beta'}, \\ p^{tr,l} &= d^l \frac{\beta}{1 - \beta'}, \\ (R^{f1,h})^{-1} &= \beta \frac{1}{2} \left(1 + \frac{d^h}{d^l} \right), \\ (R^{f1,l})^{-1} &= \beta \frac{1}{2} \left(1 + \frac{d^l}{d^h} \right). \end{aligned}$$

In state $i = h, l$, the expected return on a tree equals

$$\frac{1}{2} \frac{d^h + p^{tr,h} + d^l + p^{tr,l}}{p^{tr,i}} = \frac{1}{2\beta} \frac{d^h + d^l}{d^i}.$$

The excess return on trees over a risk-free bond thus equals

$$\frac{1}{2\beta} \frac{d^h + d^l}{d^i} - \frac{1}{\beta \frac{1}{2} \left(1 + \frac{d^i}{d^j} \right)} = \frac{1}{2\beta} \frac{(d^h - d^l)^2}{d^i (d^h + d^l)},$$

where j denotes the state currently not realized. Note that the excess return is strictly positive whenever $d^h \neq d^l$ and it is higher when d^i is low.

The capital stock in an economy may be viewed as a fruit yielding tree and we may thus associate the tree price with a broad measure of stock prices. According to that interpretation the excess return on the tree equals the expected *equity premium*.

5.6 Bibliographic Notes

The CAPM is due to Sharpe (1964), Lintner (1965), and Mossin (1966); the C-CAPM is due to Lucas (1978) and Breeden (1979). For introductions to financial economics and asset pricing, see LeRoy and Werner (2014) and Cochrane (2001). Tirole (1982) proves that bubbles cannot arise in a rational expectations equilibrium with finitely many investors. Tirole (1985) analyzes bubbles in overlapping generations economies. The model of asset prices in an endowment economy is due to Lucas (1978). Mehra and Prescott (1985) analyze implications for the equity premium; see also Weil (1989).

Beyond the material covered in the chapter, Modigliani and Miller (1958) show that the value of a firm is independent of its liability structure, a consequence of “value additivity.” Incentive problems and other frictions undermine the result, see Tirole (2006). Magill and Quinzii (1996) cover equilibrium in economies with incomplete financial markets and heterogeneous agents.

Chapter 6

Labor Supply, Growth, and Business Cycles

We have assumed so far that households supply labor at no cost and thus, that labor supply effectively is exogenous. To endogenize labor supply, we now introduce an opportunity cost of working: foregone utility from consuming leisure. We first consider the problem of a household that chooses how much to work (and save). Subsequently, we embed the household's choice in extensions of the representative agent model analyzed in section 3.1. The first extension focuses on growth and the second on business cycles. Throughout, we assume that households have a time endowment of one unit per period.

6.1 Goods Versus Leisure Consumption

6.1.1 One Period

Intensive Margin Consider first a static setting. Letting x denote leisure and thus, $1 - x$ time spent working, the program of a household reads

$$\max_{c,x} u(c,x) \text{ s.t. } c = w(1 - x),$$

where w denotes the wage. Utility depends on the consumption of goods, c , and leisure. We assume that u is strictly increasing and concave in both arguments. Note that the wage represents the price of leisure relative to goods consumption: A marginal reduction of leisure affords an increase of goods consumption w times its size. This is most evident when we express the budget constraint as

$$\frac{c}{w} + x = 1,$$

equating expenditures for goods and leisure consumption with wealth, all expressed in terms of time.

An interior optimal choice of (c, x) satisfies the first-order condition

$$u_x(c, x) = u_c(c, x)w.$$

That is, the household equates the relative price of leisure and goods consumption with the marginal rate of substitution, $u_x(c, x)/u_c(c, x)$. Combined with the budget constraint, the first-order condition yields the solution to the household's program.

A wage change alters both the household's wealth and the price of leisure relative to goods consumption. Accordingly, it induces income and substitution effects. The *compensated (Hicksian)* labor supply elasticity equals the elasticity of $1 - x$ with respect to w when utility is held constant. In contrast, the *Frisch* labor supply elasticity equals the elasticity of $1 - x$ with respect to w for a fixed marginal utility of wealth.

To derive the Frisch elasticity, assume that

$$u(c, x) = \begin{cases} \bar{u}(c) + \gamma \frac{x^{1-\varphi} - 1}{1-\varphi}, & \varphi > 0, \varphi \neq 1 \\ \bar{u}(c) + \gamma \ln(x), & \varphi = 1 \end{cases},$$

for some strictly increasing and concave \bar{u} . Letting λ denote the multiplier associated with the budget constraint—the marginal utility of wealth—the household's optimal choice of x satisfies

$$\gamma x^{-\varphi} = \lambda w.$$

The Frisch elasticity of x with respect to w thus equals $d \ln(x) / d \ln(w) = -\varphi^{-1}$ (holding λ constant) and the Frisch elasticity of labor supply equals $x / (1 - x) \varphi^{-1}$.

Extensive Margin Suppose next that workers may only work either a fraction $h > 0$ of their time or not at all that is, the labor supply choice occurs at the *extensive* rather than the intensive margin. The aggregate labor supply elasticity then can be substantially higher than suggested by the preference parameter φ .

To see this, consider a household with a continuum of identical members. Each member works h "hours" with probability p and does not work with probability $1 - p$. Aggregate labor supply therefore equals ph . Letting $\bar{x} \equiv 1 - h$, the household solves

$$\max_{p, c^0, c^1} p\bar{u}(c^1) + (1-p)\bar{u}(c^0) + \gamma \frac{p\bar{x}^{1-\varphi} + (1-p) \cdot 1 - 1}{1-\varphi} \quad \text{s.t.} \quad pc^1 + (1-p)c^0 = p(1-\bar{x})w,$$

where c^1 and c^0 denote consumption of working and non-working household members, respectively. Risk sharing implies $c^1 = c^0 = c$ (see section 4.2). Letting

$$\bar{\gamma} \equiv \gamma \frac{1 - \bar{x}^{1-\varphi}}{1-\varphi} > 0, \quad \bar{w} \equiv (1 - \bar{x})w.$$

the program can then be expressed as

$$\max_{p, c} \bar{u}(c) - \bar{\gamma}p \quad \text{s.t.} \quad c = p\bar{w}$$

which corresponds to the intensive margin program of a household with preference parameter $\varphi = 0$ that faces wage \bar{w} . We conclude that the household labor supply elasticity at the extensive margin equals infinity.

6.1.2 More Periods

With two (and similarly, with more) periods, the household's choice at the intensive margin solves

$$\max_{c_0, c_1, x_0, x_1} u(c_0, x_0) + \beta u(c_1, x_1) \quad \text{s.t.} \quad c_0 + \frac{c_1}{R_1} + w_0 x_0 + \frac{w_1 x_1}{R_1} = w_0 + \frac{w_1}{R_1}$$

and the first-order conditions yield

$$u_c(c_0, x_0) = \beta R_1 u_c(c_1, x_1) \quad \text{and} \quad u_x(c_t, x_t) = u_c(c_t, x_t) w_t.$$

The intertemporal first-order condition can also be expressed as

$$u_x(c_0, x_0) = \beta R_1 \frac{w_0}{w_1} u_x(c_1, x_1).$$

This condition equates the relative price of leisure in the first and second period, $R_1 w_0 / w_1$, with the corresponding marginal rate of substitution. If u is additively separable then leisure consumption rises and labor supply falls over time if and only if $\beta R_1 w_0 / w_1 > 1$.

Suppose that

$$u(c, x) = \ln(c) + \frac{x^{1-\varphi} - 1}{1-\varphi}, \quad \varphi > 0, \varphi \neq 1.$$

Substituting from the intra- and intertemporal first-order conditions into the budget constraint and letting $w \equiv w_1 / w_0$, we arrive at

$$x_0^\varphi (1 + \beta) + x_0 \left(1 + \frac{(\beta R_1 / w)^{1/\varphi} w}{R_1} \right) = 1 + \frac{w}{R_1}.$$

The terms on the left-hand side represent outlays for goods and leisure consumption, expressed in terms of the numeraire, leisure in the first period. The terms on the right-hand side represent wealth.

Figure 6.1 plots (c_0, c_1, x_0, x_1) against φ under the assumption that $w > 1$ and $\beta R_1 = 1$. The figure illustrates that a household with a lower intertemporal elasticity of substitution, $1/\varphi$, chooses a smoother leisure profile over time. At the same time, a higher value of φ increases the marginal utility of leisure such that the household raises leisure relative to goods consumption. Goods consumption is constant over time because $\beta R_1 = 1$; in contrast, leisure consumption decreases over time because $\beta R_1 w_0 / w_1 < 1$.

6.2 Growth

6.2.1 Exogenous Growth

We introduce two elements into the environment considered in section 3.1. First, we assume that the representative household values leisure. Second, we allow for exogenous productivity as an additional determinant of output. While we maintain the

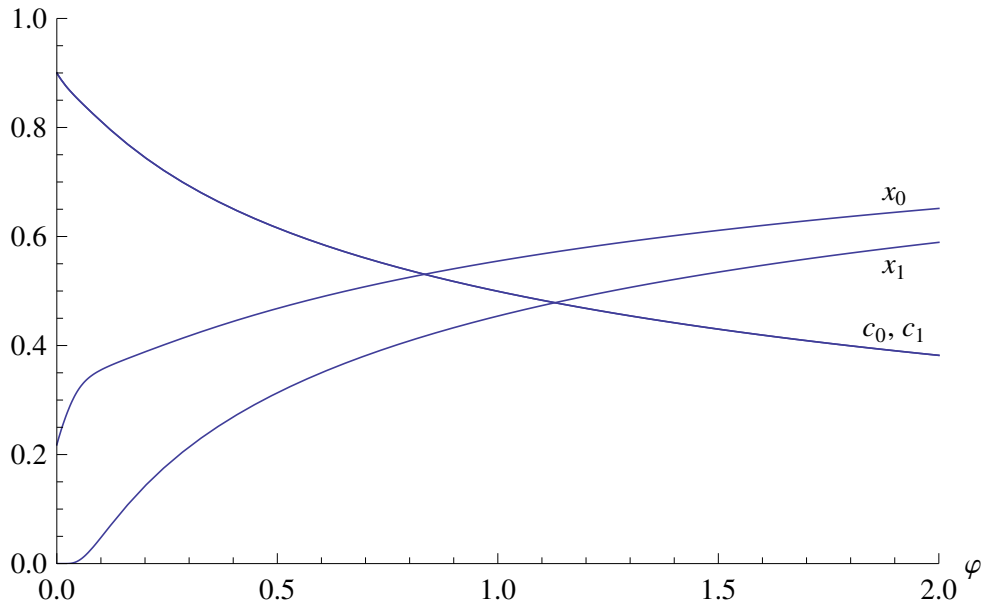


Figure 6.1: Goods and leisure consumption: (c_0, c_1, x_0, x_1) for different values of φ when $w_1 > w_0$ and $\beta R_1 = 1$. (Parameter values: $w_0 = 0.9, w_1 = 1.1, \beta = 0.98$.)

assumption of CRTS in capital and labor, we adopt a very general specification of productivity growth: We assume that per-capita output, y_t , depends on time, in addition to the per-capita capital stock, k_t , and per-capita labor input, $1 - x_t$,

$$y_t = \tilde{f}(k_t, 1 - x_t, t).$$

Conditional on its third argument the production function \tilde{f} satisfies the Inada conditions and exhibits decreasing marginal products. We also allow for exogenous gross population growth at rate ν .

In this modified environment, the first welfare theorem continues to apply. The decentralized equilibrium therefore solves the planner problem

$$\begin{aligned} \max_{\{c_t, x_t, k_{t+1}\}_{t \geq 0}} & \sum_{t=0}^{\infty} \beta^t \nu^t u(c_t, x_t) & (6.1) \\ \text{s.t.} & \nu k_{t+1} = k_t(1 - \delta) + \tilde{f}(k_t, 1 - x_t, t) - c_t, \quad k_0 \text{ given, } k_{t+1} \geq 0. \end{aligned}$$

Before deriving the optimality (and thus, equilibrium) conditions, we analyze what types of technology and preferences are consistent with a *balanced growth path* (BGP), that is a path along which all variables grow at constant (but possibly different) rates, and where c_t, k_t , and y_t all are strictly positive.

Suppose that the economy starts to grow along a BGP at date $t = T$ and let γ_z denote the gross growth rate of a generic variable z along the BGP. Note that we must have $\gamma_x = 1$ since the time endowment is bounded. Dividing the resource constraint

in program (6.1) at date $t > T$ by γ_k^{t-T} and rearranging yields

$$k_T(\nu\gamma_k - 1 + \delta) = y_T(\gamma_y/\gamma_k)^{t-T} - c_T(\gamma_c/\gamma_k)^{t-T}.$$

Since the left-hand side of this equality is independent of t the right-hand side must be time independent as well. Moreover, since $y_T, k_T, c_T > 0$, this implies $\gamma_y = \gamma_k = \gamma_c$.

Denoting the common gross growth rate of per-capita output, capital and consumption by γ we thus have established

$$k_T(\nu\gamma - 1 + \delta) = y_T - c_T \quad \text{and} \quad y_T\gamma^{t-T} = \tilde{f}(k_T\gamma^{t-T}, 1 - x_T, t).$$

The latter equality and CRTS imply

$$\tilde{f}(k_T, 1 - x_T, T) = y_T = \frac{1}{\gamma^{t-T}}\tilde{f}(k_T\gamma^{t-T}, 1 - x_T, t) = \tilde{f}\left(k_T, \frac{1 - x_T}{\gamma^{t-T}}, t\right).$$

Comparing the left- and right-most expressions, we conclude that the only form of technological progress consistent with a BGP is *labor augmenting* technological progress at the gross growth rate γ .

Up to some normalization of the initial level of technology, output per capita thus depends on the per-capita capital stock and per-capita labor supply in *efficiency units*, $(1 - x_t)\gamma^t$:

$$y_t = f(k_t, (1 - x_t)\gamma^t),$$

where f denotes the neoclassical production function considered in section 3.1. In the special case of a Cobb-Douglas production function, labor augmenting (or “Harrod-neutral”) technological progress is isomorphic to progress that is capital augmenting (“Solow-neutral”) or multiplying $f(k, 1 - x)$ (“Hicks-neutral”).

BGP dynamics also impose restrictions on preferences. To see this, consider the Euler equation and intratemporal first-order condition implied by program (6.1). These conditions read

$$\begin{aligned} \frac{u_c(c_t, x_t)}{\beta u_c(c_{t+1}, x_{t+1})} &= 1 - \delta + f_K(k_{t+1}, (1 - x_{t+1})\gamma^{t+1}), \\ \frac{u_x(c_t, x_t)}{u_c(c_t, x_t)\gamma^t} &= f_L(k_t, (1 - x_t)\gamma^t). \end{aligned}$$

Due to CRTS, the terms on the right-hand sides—which in decentralized equilibrium correspond to the gross interest rate, R_{t+1} , and the normalized wage, w_t/γ^t —are constant along a BGP. Consistency therefore requires that the elasticity of u_c with respect to c_t is constant and that u_c/u_x falls at the same rate as c_t grows. These two requirements imply

$$u(c, x) = \begin{cases} \frac{c^{1-\sigma}v(x)-1}{1-\sigma}, & \sigma > 0, \sigma \neq 1 \\ \ln(c) + v(x), & \sigma = 1 \end{cases},$$

where the function v needs to satisfy additional conditions to guarantee that u is increasing and concave.

To gain intuition for the preference restrictions, note that along a BGP consumption and the wage grow at the same rate such that the intratemporal first-order condition takes the form

$$u_c(w_T \gamma^{t-T} \bar{\zeta}, x_T) w_T \gamma^{t-T} = u_x(w_T \gamma^{t-T} \bar{\zeta}, x_T)$$

for all $t > T$ where $\bar{\zeta} > 0$ denotes some constant. To satisfy this condition, preferences must give rise to wage induced income and substitution effects on leisure consumption that cancel each other out in a static environment (see subsection 6.1.1). The BGP restrictions then also generate Kaldor's (1961) "stylized facts" including a constant capital-output ratio, constant wage growth and interest rates, and constant factor shares in national income.

A final condition that we need to impose on the primitives concerns the level of productivity growth: it must not be too high because otherwise, the objective is unbounded. To see this, note that

$$\sum_{t=T}^{\infty} \beta^{t-T} v^{t-T} \frac{(c_T \gamma^{t-T})^{1-\sigma} v(x_T) - 1}{1-\sigma} = \sum_{t=T}^{\infty} (\beta v \gamma^{1-\sigma})^{t-T} \chi_1 - \sum_{t=T}^{\infty} (\beta v)^{t-T} \chi_2,$$

where χ_1, χ_2 denote constants. Boundedness requires $\beta v \gamma^{1-\sigma} < 1$.

It is useful to express program (6.1) in terms of stationary that is, detrended variables. To this end, we normalize all variables except leisure by the cumulative BGP growth rate, γ^t . Letting a "bar" denote normalized variables, this yields

$$\begin{aligned} \max_{\{\bar{c}_t, x_t, \bar{k}_{t+1}\}_{t \geq 0}} \quad & \sum_{t=0}^{\infty} \beta^t v^t \gamma^{t(1-\sigma)} u(\bar{c}_t, x_t) \\ \text{s.t.} \quad & v \gamma \bar{k}_{t+1} = \bar{k}_t (1 - \delta) + f(\bar{k}_t, 1 - x_t) - \bar{c}_t, \quad \bar{k}_0 \text{ given}, \quad \bar{k}_{t+1} \geq 0, \end{aligned} \quad (6.2)$$

where u satisfies the restrictions discussed above. Defining $\beta^* \equiv \beta v \gamma^{1-\sigma}$, the first-order conditions simplify to

$$u_x(\bar{c}_t, x_t) = f_L(\bar{k}_t, 1 - x_t) u_c(\bar{c}_t, x_t), \quad (6.3)$$

$$v \gamma u_c(\bar{c}_t, x_t) = \beta^* (1 - \delta + f_K(\bar{k}_{t+1}, 1 - x_{t+1})) u_c(\bar{c}_{t+1}, x_{t+1}). \quad (6.4)$$

Condition (6.3) is equivalent to the intratemporal first-order condition in the decentralized equilibrium,

$$u_x(c_t, x_t) = f_L(k_t, (1 - x_t) \gamma^t) \gamma^t u_c(c_t, x_t) = w_t u_c(c_t, x_t),$$

and condition (6.4) is equivalent to the standard Euler equation

$$u_c(c_t, x_t) = \beta (1 - \delta + f_K(\bar{k}_{t+1}, 1 - x_{t+1})) u_c(c_{t+1}, x_{t+1}) = \beta R_{t+1} u_c(c_{t+1}, x_{t+1}).$$

Since along the BGP, per-capita consumption grows at the gross rate γ and thus, marginal utility at rate $\gamma^{-\sigma}$, the Euler equation implies that the gross *interest rate* satisfies

$$R = \gamma^\sigma / \beta.$$

When $\gamma \neq 1$, the interest rate and thus, the capital-labor ratio reflect the curvature of preferences and the growth rate, in addition to the psychological discount factor. Intuitively, in a growing (or shrinking) economy, per-capita consumption grows (or shrinks) as well; since the substitution effect of the interest rate on consumption depends on the curvature of preferences the result follows. With positive growth, more patience (higher β) or more willingness to intertemporally substitute (lower σ) imply lower interest rates and a higher capital intensity.

6.2.2 Endogenous Growth

Long-run per-capita growth in the model of the previous subsection reflects exogenous technological progress. The model explains why per-capita output, investment and consumption grow at the same rate as technology, but it does not explain why technology and thus, the economy grow.

Underlying the model's inability to endogenously generate sustained per-capita growth is the Inada condition: As the capital intensity rises, the marginal product of capital declines and the incentive to further accumulate falls. We now relax this condition and show that models with a suitably modified production function endogenously generate sustained per-capita growth. Throughout the subsection, we abstract from technological advances and assume that the population size is constant, $\gamma = \nu = 1$. We also abstract from leisure, $x_t = 0$.

Ak Technology

To make the point, consider first an extreme production function,

$$f(K, L) = AK, \quad A > 0.$$

Function f exhibits CRTS but does not feature decreasing marginal products; in fact, the marginal product of capital is constant. With this *Ak technology* the resource constraint reads

$$k_{t+1} = k_t(1 - \delta) + Ak_t - c_t,$$

wages equal zero, and the interest rate is constant at value $R_t = 1 - \delta + A$. The Euler equation therefore implies that marginal utility grows at a constant rate. With CIES preferences,

$$\left(\frac{c_{t+1}}{c_t}\right)^\sigma = \beta(1 - \delta + A),$$

and from the resource constraint (and the transversality condition), the capital stock grows at this rate as well. Dividing the resource constraint by k_t and substituting the expression for the growth rate yields the equilibrium initial consumption level, c_0 , for a given initial capital stock, k_0 :

$$[\beta(1 - \delta + A)]^{\frac{1}{\sigma}} = 1 - \delta + A - \frac{c_0}{k_0}.$$

The consumption-capital ratio c_0/k_0 is maintained forever—the economy immediately reaches the BGP.

The economy exhibits sustained per-capita growth if $\beta R > 1$ but household utility only is well defined if $\sum_{t=0}^{\infty} \beta^t c_t^{1-\sigma} < \infty$, that is if $\beta(c_{t+1}/c_t)^{1-\sigma} < 1$. Both conditions are satisfied if

$$\beta(1 - \delta + A) > 1 > \beta(1 - \delta + A)^{1-\sigma}.$$

Note that both technology (A and δ) and preferences (β and σ) affect the equilibrium growth rate, in contrast to the situation with a neoclassical production function. If policy affected the interest rate received by investors (for example by taxing or subsidizing capital income) it would also affect the growth rate.

Two Sectors

One unappealing feature of the Ak model is that labor income equals zero. A two-sector version of the model where the first sector operates a neoclassical technology to produce the consumption good while the second produces investment goods with an Ak technology, remedies this problem. The model makes clear that endogenous growth does not require a linear technology in all sectors; it suffices if the marginal product of the production function for the accumulated factor of production (capital) is bounded from below.

Let k_t^c denote the capital stock employed in the production of consumption goods; the remaining capital stock, $k_t - k_t^c$, is used to produce investment goods. We assume a Cobb-Douglas production function in the consumption goods sector, $c_t = (k_t^c)^\alpha$, where $\alpha \in (0, 1)$ denotes the capital share and labor is normalized to unity. Since the environment satisfies the conditions of the first welfare theorem we can solve the social planner problem to characterize equilibrium. This program reads

$$\begin{aligned} \max_{\{c_t, k_t^c, k_{t+1}\}_{t \geq 0}} \quad & \sum_{t=0}^{\infty} \beta^t u((k_t^c)^\alpha) \\ \text{s.t.} \quad & k_{t+1} = k_t(1 - \delta) + A(k_t - k_t^c), \quad k_0 \text{ given, } k_{t+1} \geq 0, k_t \geq k_t^c \geq 0, \end{aligned}$$

and the first-order conditions with respect to k_t^c and k_{t+1} yield

$$u'(c_t)\alpha(k_t^c)^{\alpha-1} = u'(c_{t+1})\alpha(k_{t+1}^c)^{\alpha-1}\beta(1 - \delta + A).$$

Along a BGP, k_t^c and k_t grow at the same rate. Using this fact as well as the production function in the consumption goods sector and imposing CIES preferences, we find the following relation between the BGP gross growth rates of consumption and capital, γ_c and γ_k respectively:

$$\gamma_c = \gamma_k^\alpha = [\beta(1 - \delta + A)]^{\frac{\alpha}{1-\alpha+\alpha\sigma}}.$$

Consumption grows more slowly than the capital stock because the production function in the consumption goods sector exhibits decreasing returns to scale (holding labor

fixed). In the decentralized equilibrium the growth differential is reflected in a trend increase of the price of consumption relative to investment goods; this price differential renders investors indifferent between the two sectors although the marginal product of capital in the consumption goods sector faces a secular decline.

Externalities

Endogenous growth does not require a production function for the accumulated factor of production whose marginal product is bounded from below at the level of an individual firm; an aggregate production function with this property suffices, and technological spillovers may render the former different from the latter.

To see this, consider a one-sector model where the representative firm operates the technology

$$y_t = A_t f(k_t, 1) = A_t k_t^\alpha.$$

Labor is normalized to unity, the capital share $\alpha \in (0, 1)$, and A_t denotes productivity which households and firms (rationally) take as given.

Suppose that productivity depends positively on the average capital stock in the economy, which in equilibrium equals the capital stock owned by each individual household,

$$A_t = A k_t^{1-\alpha}.$$

In equilibrium, per-capita output then is a linear function of the capital-labor ratio, $y_t = A k_t$, while at the level of an individual firm, the production function is neoclassical.

Since investment by an individual firm increases the productivity of all firms—it generates a positive externality—the conditions of the first welfare theorem are violated. To characterize the decentralized equilibrium, we therefore need to derive the household and firm optimality conditions rather than those of a social planner. The program of the representative household who takes productivity as given reads

$$\begin{aligned} \max_{\{c_t, k_{t+1}\}_{t \geq 0}} \quad & \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t.} \quad & k_{t+1} = k_t(1 - \delta) + A_t k_t^\alpha - c_t, \quad k_0 \text{ given}, \quad k_{t+1} \geq 0. \end{aligned}$$

Assuming CIES preferences, the first-order conditions of this program combined with the relation between productivity and the average capital stock reduce to

$$\left(\frac{c_{t+1}}{c_t} \right)^\sigma = \beta(1 - \delta + \alpha A).$$

Wages and interest rates are determined by firms' marginal products, taking productivity as given.

Under parameter conditions similar to those discussed earlier, equilibrium growth is strictly positive in the long run and the objective function is bounded. In contrast to the growth models considered so far, however, equilibrium growth is inefficiently low. Unlike individual households, a social planner would solve the program subject to the additional constraint that $A_t = A k_t^{1-\alpha}$. As a consequence, the social planner would implement an allocation with a higher growth rate than in competitive equilibrium.

6.3 Business Cycles

6.3.1 Real Business Cycles

To study “real,” that is technology-driven business cycles, we modify the neoclassical growth model analyzed in subsection 6.2.1 in one respect: We assume that technology does not only grow deterministically, at gross rate γ , but may also fluctuate temporarily and stochastically. Specifically, we assume that per-capita capital stock, k_t , and per-capita labor supply, $1 - x_t$, generate per-capita output

$$y_t = f(k_t, (1 - x_t)\gamma^t) \cdot A_t,$$

where A_t —the new model element—stochastically fluctuates around a mean value of one. We assume that in period t and history ϵ^t , the relative deviation of $A_t(\epsilon^t)$ from its mean, $\hat{A}_t(\epsilon^t) \equiv A_t(\epsilon^t) - 1$, is governed by the first-order stochastic difference equation

$$\hat{A}_t(\epsilon^t) = \rho_A \hat{A}_{t-1}(\epsilon^{t-1}) + \iota_t(\epsilon^t), \quad 0 \leq \rho_A < 1,$$

where $\iota_t(\epsilon^t)$ is i.i.d. with mean zero. We maintain the assumption that f exhibits CRTS.

The fundamental theorems of welfare economics apply. We can therefore characterize the equilibrium by solving the planner’s problem, corresponding to program (6.2) augmented by the stochastic productivity term. To simplify the notation, we adopt a recursive formulation and let (\bar{k}_o, A_o) denote the state, $(\bar{c}_o, x_o, \bar{k}_+)$ the control, and ι_+ the shock in the subsequent period. The Bellman equation reads

$$\begin{aligned} V(\bar{k}_o, A_o) &= \max_{\bar{c}_o, x_o, \bar{k}_+} \{u(\bar{c}_o, x_o) + \beta^* \mathbb{E} [V(\bar{k}_+, A_+) | A_o]\} & (6.5) \\ \text{s.t.} \quad v\gamma\bar{k}_+ &= \bar{k}_o(1 - \delta) + f(\bar{k}_o, 1 - x_o)A_o - \bar{c}_o, \\ A_+ &= 1 + \rho_A(A_o - 1) + \iota_+, \end{aligned}$$

where $\beta^* \equiv \beta v \gamma^{1-\sigma}$ and the controls are bounded. The first-order conditions and envelope condition reduce to

$$u_x(\bar{c}_o, x_o) = f_L(\bar{k}_o, 1 - x_o)A_o u_c(\bar{c}_o, x_o), \quad (6.6)$$

$$v\gamma u_c(\bar{c}_o, x_o) = \beta^* \mathbb{E}[\{1 - \delta + f_K(\bar{k}_+, 1 - x_+)A_+\} u_c(\bar{c}_+, x_+) | A_o], \quad (6.7)$$

where (\bar{c}_+, x_+) denote optimal control choices in the subsequent period. These equilibrium conditions differ from (6.3) and (6.4) only insofar as marginal products are augmented by the corresponding A terms and future outcomes are weighted by their respective conditional probabilities.

In the special case with a Cobb-Douglas production function, full depreciation, and a utility function that is logarithmic in consumption and additively separable, analytical solutions for $(\bar{c}_o, x_o, \bar{k}_+)$ are available. To derive the equilibrium allocation under more general assumptions, we may solve the Bellman equation numerically or resort to an approximate solution based on the linearized equilibrium conditions.

For the latter strategy, consider first the deterministic BGP of the economy which results when A_t always equals one. The steady-state values of the detrended variables, (\bar{k}, \bar{c}, x) , satisfy

$$\begin{aligned} v\gamma\bar{k} &= \bar{k}(1 - \delta) + f(\bar{k}, 1 - x) - \bar{c}, \\ u_x(\bar{c}, x) &= f_L(\bar{k}, 1 - x)u_c(\bar{c}, x), \\ v\gamma u_c(\bar{c}, x) &= \beta^*(1 - \delta + f_K(\bar{k}, 1 - x))u_c(\bar{c}, x). \end{aligned}$$

Linearizing the resource constraint in program (6.5) as well as the optimality conditions (6.6) and (6.7) about the steady state values (\bar{k}, \bar{c}, x) yields a system of linear difference equations. In forming this system, we exploit the certainty equivalence property: If a non-linear function h is linearized about the value z then $\mathbb{E}_o[h(z_+)] \approx \mathbb{E}_o[h(z) + h'(z) \cdot (z_+ - z)] = h(z) + h'(z) \cdot (\mathbb{E}_o[z_+] - z)$; that is, in the linearized system, only the conditional mean of random variables is relevant. Using this property and substituting the linearized intratemporal first-order condition into the other two linearized equilibrium conditions, we arrive at (switching to sequence notation)

$$\begin{aligned} \begin{bmatrix} \hat{k}_{t+1}(\epsilon^t) \\ \mathbb{E}_t[\hat{c}_{t+1}(\epsilon^{t+1})] \end{bmatrix} &= M \begin{bmatrix} \hat{k}_t(\epsilon^{t-1}) \\ \hat{c}_t(\epsilon^t) \end{bmatrix} + N_1 \mathbb{E}_t[\hat{A}_{t+1}(\epsilon^{t+1})] + N_0 \hat{A}_t(\epsilon^t) \\ &= M \begin{bmatrix} \hat{k}_t(\epsilon^{t-1}) \\ \hat{c}_t(\epsilon^t) \end{bmatrix} + N \hat{A}_t(\epsilon^t). \end{aligned}$$

Here, a circumflex denotes relative deviations from the steady-state value (e.g., $\hat{c}_t \equiv (\bar{c}_t - \bar{c})/\bar{c}$), the elements of the (2×2) matrices M , N_0 , and N_1 contain parameters and functions evaluated at the steady-state values, and $N \equiv \rho_A N_1 + N_0$.

This system with one predetermined (capital) and one non-predetermined endogenous variable (consumption) differs twofold from the system analyzed previously, in the context of the deterministic representative agent model (see subsection 3.1.7). First, matrices M and N do not only reflect the linearized resource constraint and Euler equation in general equilibrium but they also incorporate the intratemporal first-order condition that relates the leisure choice to consumption, the capital stock and productivity. Second, the presence of temporary productivity variation introduces an exogenous shock process. As noted before, stochasticity of the productivity shock does not introduce any additional complication because of certainty equivalence.

We solve the system using essentially the same approach as in the deterministic environment. Since the matrix M has one stable and one unstable eigenvalue, the equilibrium value $\hat{c}_t(\epsilon^t)$ (conditional on $\hat{k}_t(\epsilon^{t-1})$ and $\hat{A}_t(\epsilon^t)$) is uniquely determined by the requirement that the difference equation system generates paths for expected consumption and capital that converge to their steady-state values, (\bar{c}, \bar{k}) . Moreover, given $\hat{k}_t(\epsilon^{t-1})$, $\hat{A}_t(\epsilon^t)$, and $\hat{c}_t(\epsilon^t)$, the intratemporal first-order condition uniquely determines $x_t(\epsilon^t)$. Appendix B.5 offers a discussion.

Figures 6.2–6.4 illustrate the response of the model economy to a productivity shock at date $t = 3$. We assume a Cobb-Douglas production function with capital share 0.3 and a depreciation rate of 5 percent, a discount factor $\beta = 0.98$, and CIES preferences

of the form

$$u(c, x) = \begin{cases} \ln(c) + \frac{x^{1-\varphi}-1}{1-\varphi}, & \varphi > 0, \varphi \neq 1 \\ \ln(c) + \ln(x), & \varphi = 1 \end{cases}.$$

We compare three scenarios, distinguished by the autocorrelation of the technology parameter, ρ_A , and the willingness of households to intertemporally substitute leisure, $1/\varphi$.

Figure 6.2 illustrates the response to a temporary technology shock, $\rho_A = 0$, when the elasticity of substitution equals unity, $\varphi = 1$. Technology improves at date $t = 3$, $\iota_3 > 0$, and is never shocked again, $\iota_t = 0, t \neq 3$; it immediately reverts to its steady-state value A at date $t = 4$ because $\rho_A = 0$.

In equilibrium, all endogenous variables except for the predetermined capital stock contemporaneously respond to the productivity shock. Wages and consumption rise relative to their steady-state values and eventually revert back to the latter. Leisure falls on impact before rising in the subsequent period and similarly embarking on a path back to its steady-state value. The interest rate responds inversely, and the capital stock increases at date $t = 4$ before converging back to steady state.

To interpret these developments, consider a household in decentralized equilibrium. Following the technology shock, this household faces an increased wage and anticipates higher wages and lower interest rates in the future. This renders the household wealthier, inducing higher consumption, while at the same time the lower interest rates induce a substitution effect towards present consumption (see the Euler equation (6.7)). The high wage at date $t = 3$ also induces a strong substitution effect towards goods relative to leisure consumption (see condition (6.6)), stimulating labor supply. In subsequent periods, the wealth effect on leisure consumption dominates this substitution effect, leading to an increase of leisure consumption relative to steady state.

The higher output due to increased productivity and stronger labor supply at date $t = 3$ is not fully consumed. Part of it is saved and invested, generating a higher capital stock in subsequent periods. It is this higher capital stock from date $t = 4$ onward that keeps wages persistently elevated and interest rates subdued. From date $t = 4$ onward, negative net investment generates resources for consumption. Capital accumulation at date $t = 3$ and dissipation starting at date $t = 4$ thus allow to smooth aggregate consumption.

By construction, the equilibrium dynamics satisfy the resource constraint and the optimality conditions (6.6) and (6.7) at all times. Note, however, that the Euler equation prescribes equality of the marginal rate of substitution and the marginal rate of transformation only in expected terms. At the time of the shock, $t = 3$, expected (average) and realized values differ. Accordingly, the realized interest and consumption growth rates do not satisfy the deterministic version of the Euler equation. Related, equilibrium goods and leisure consumption in the shock period are determined in a forward looking way, by the requirement that their choice places the economy on the saddle path subject to the productivity shock sequence.

Figure 6.3 illustrates the response to the same shock at date $t = 3$, $\iota_3 > 0$, but under the assumption that such a shock is highly persistent, $\rho_A = 0.99$. We keep the elasticity

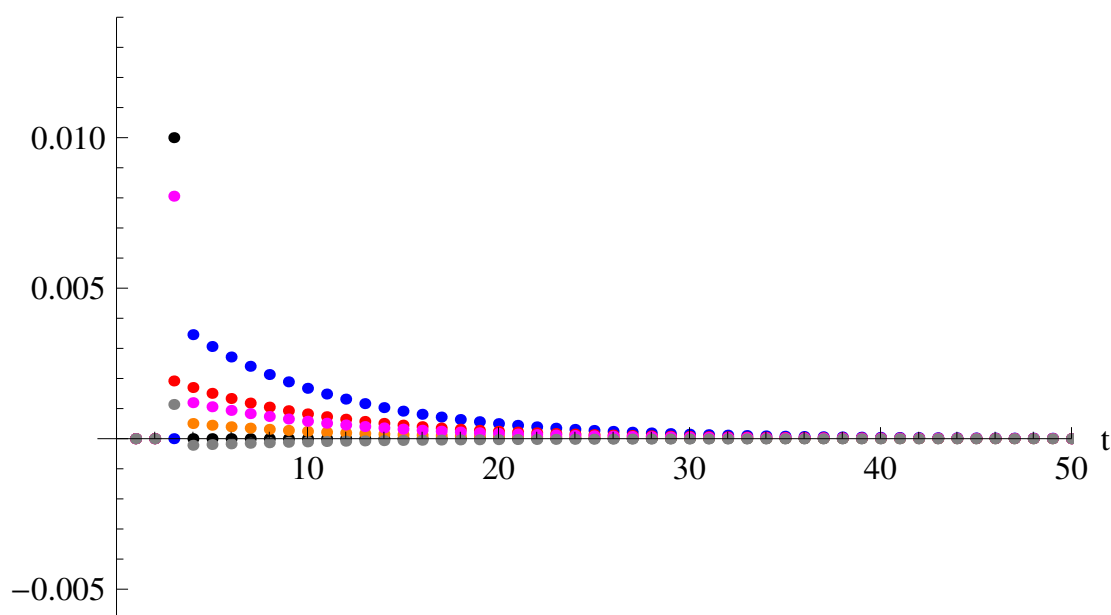


Figure 6.2: Effects of a productivity shock when $\rho_A = 0$ and $\varphi = 1$. $\hat{A}_t, \hat{k}_t, \hat{c}_t, \hat{x}_t, \hat{w}_t$, and \hat{R}_t are indicated in black, blue, red, orange, magenta and gray, respectively.

of substitution φ unchanged.

The persistent technological improvement gives rise to a similarly persistent rise in wages. Capital is accumulated during more than just one period because the marginal product of capital is elevated until date $t = 17$ (reflected in the interest rate), inducing households to delay consumption in spite of the strong wealth effect. Because of the long-lasting rise of wages, leisure consumption is persistently depressed and accordingly, labor supply stimulated.

Finally, figure 6.4 illustrates the response to the same shock under the assumption that it is persistent, $\rho_A = 0.9$, and the elasticity of substitution twice as high as before, $\varphi = 0.5$. The lower persistence gives rise to a faster convergence of the endogenous variables, and the higher elasticity implies a stronger labor supply response to the shock.

6.3.2 “Sunspot”-Driven Business Cycles

In the Real Business Cycle model, fluctuations are driven by exogenous shocks to productivity. Each level of productivity and its expected future path are associated with a unique set of equilibrium paths for the endogenous variables. The uniqueness reflects the saddle-path property of the dynamic system: Conditional on the predetermined capital stock, only one initial value for consumption and leisure is consistent with the equilibrium conditions including the requirement that system dynamics be stable. In turn, the saddle-path property reflects the fact that the number of unstable eigenvalues in the dynamic system equals the number of non-predetermined variables.

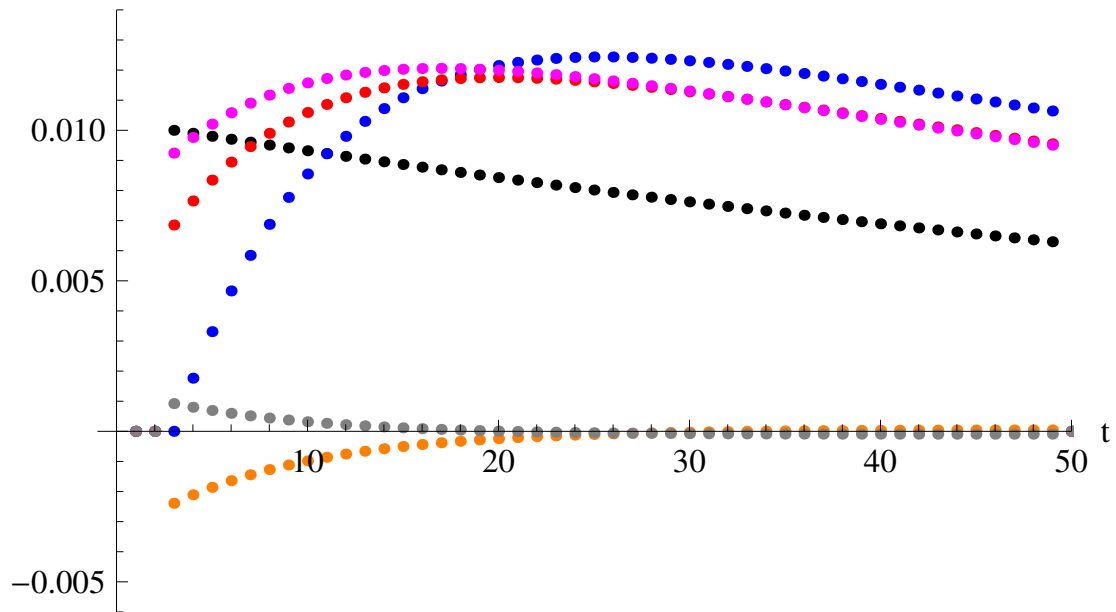


Figure 6.3: Effects of a productivity shock when $\rho_A = 0.99$ and $\varphi = 1$. $\hat{A}_t, \hat{k}_t, \hat{c}_t, \hat{x}_t, \hat{w}_t,$ and \hat{R}_t are indicated in black, blue, red, orange, magenta and gray, respectively.

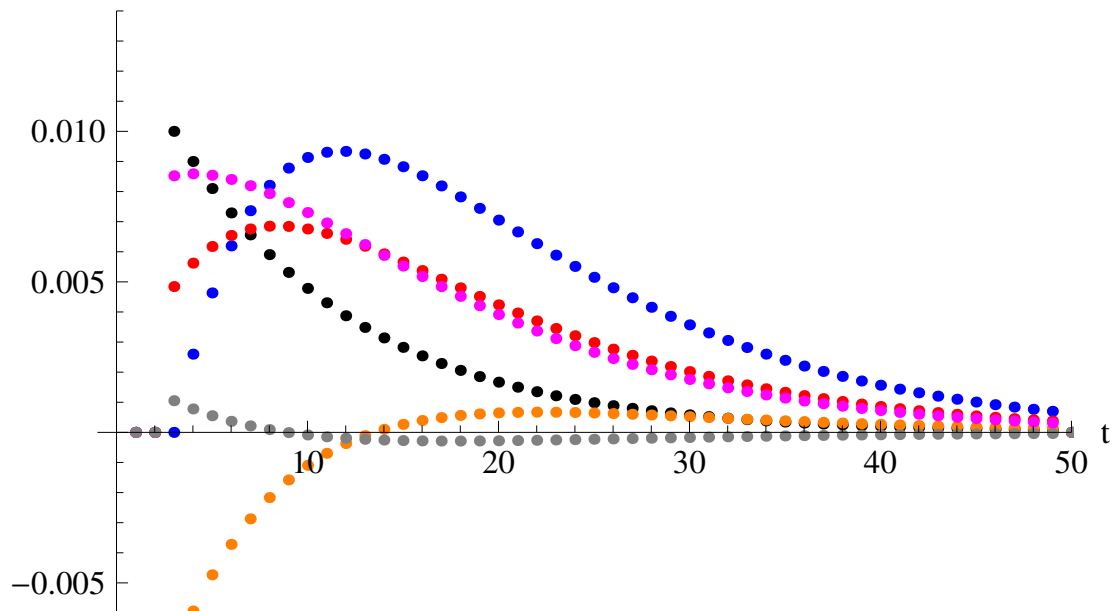


Figure 6.4: Effects of a productivity shock when $\rho_A = 0.9$ and $\varphi = 0.5$. $\hat{A}_t, \hat{k}_t, \hat{c}_t, \hat{x}_t, \hat{w}_t,$ and \hat{R}_t are indicated in black, blue, red, orange, magenta and gray, respectively.

Consider now a different dynamic system where the number of unstable eigenvalues is strictly smaller than the number of non-predetermined variables. Conditional on the predetermined capital stock, multiple initial values for consumption and leisure then are consistent with the equilibrium conditions, see appendix B.5. As a consequence, the endogenous variables may respond not only to fundamental exogenous shocks, for instance to the level of productivity, but also to “sunspot” shocks without any fundamental bearing on the economy. We now analyze a model where this is the case.

To render the analysis as transparent as possible we assume that there are no fundamental exogenous shocks at all. If the dynamic system exhibited the saddle-path property and the capital stock equalled its BGP value, all other endogenous variables therefore would equal their BGP values as well. However, we modify the production function with the consequence that the system does not exhibit the saddle-path property (the number of non-predetermined variables exceeds the number of unstable eigenvalues by one). Conditional on the value of capital the requirement that system dynamics satisfy all equilibrium conditions including stability then leaves one degree of freedom—the initial level of consumption or leisure is arbitrary and may reflect a non-fundamental “sunspot” shock.

We assume that the production function exhibits increasing returns to scale, similar to the growth model with externalities considered in subsection 6.2.2. Specifically, we assume that per-capita output is given by

$$y_t = f(k_t, (1 - x_t)\gamma^t) \cdot A_t,$$

where A_t does not exogenously fluctuate as in the Real Business Cycle model, but instead is determined by the aggregate level of production and thus, aggregate factor inputs that each individual firm and household takes as given:

$$A_t = f(k_t, (1 - x_t)\gamma^t)^\chi, \chi \geq 0.$$

For $\chi = 0$, the model reduces to the Real Business Cycle model with a constant level of productivity.

Due to increasing returns to scale the economy does not satisfy the conditions of the welfare theorems. To characterize the decentralized equilibrium, we therefore use the first-order conditions of households and firms (that take productivity as given) in the Real Business Cycle model and replace A_t in those conditions by the expression above. Along the BGP with gross population growth rate ν , per-capita output grows at gross rate μ say. Adopting recursive notation, we can express the equilibrium conditions in terms of detrended variables as

$$\begin{aligned} \nu\mu\bar{k}_+ &= \bar{k}_o(1 - \delta) + f(\bar{k}_o, 1 - x_o)^{1+\chi} - \bar{c}_o, \\ u_x(\bar{c}_o, x_o) &= f_L(\bar{k}_o, 1 - x_o)f(\bar{k}_o, 1 - x_o)^\chi u_c(\bar{c}_o, x_o), \\ \nu\mu u_c(\bar{c}_o, x_o) &= \beta^* \mathbb{E}[\{1 - \delta + f_K(\bar{k}_+, 1 - x_+)\} u_c(\bar{c}_+, x_+)], \end{aligned}$$

where $\beta^* \equiv \beta\nu\mu^{1-\sigma}$. The unique steady-state values of the detrended variables, (\bar{k}, \bar{c}, x) ,

satisfy

$$\begin{aligned} v\mu\bar{k} &= \bar{k}(1-\delta) + f(\bar{k}, 1-x)^{1+\chi} - \bar{c}, \\ u_x(\bar{c}, x) &= f_L(\bar{k}, 1-x)f(\bar{k}, 1-x)^\chi u_c(\bar{c}, x), \\ v\mu u_c(\bar{c}, x) &= \beta^*(1-\delta + f_K(\bar{k}, 1-x)f(\bar{k}, 1-x)^\chi)u_c(\bar{c}, x). \end{aligned}$$

A Cobb-Douglas production function with capital share α implies

$$\mu \equiv \gamma \frac{(1-\alpha)(1+\chi)}{1-\alpha(1+\chi)} v \frac{\chi}{1-\alpha(1+\chi)}.$$

Note that μ reduces to γ if $\chi = 0$.

Linearizing the system of equilibrium conditions about the steady state and reducing it to a system of two difference equations in capital and consumption yields the system (switching to sequence notation)

$$\begin{bmatrix} \hat{k}_{t+1}(\epsilon^t) \\ \mathbb{E}_t[\hat{c}_{t+1}(\epsilon^{t+1})] \end{bmatrix} = M \begin{bmatrix} \hat{k}_t(\epsilon^{t-1}) \\ \hat{c}_t(\epsilon^t) \end{bmatrix}.$$

As before, a circumflex denotes relative deviations from the steady-state value (e.g., $\hat{c}_t \equiv (\bar{c}_t - \bar{c})/\bar{c}$) and the elements of the (2×2) matrix M contain parameters and functions evaluated at steady-state values.

If φ is sufficiently small and χ sufficiently large, the matrix M has two stable eigenvalues. Conditional on the predetermined capital stock the equilibrium conditions then are consistent with an arbitrary initial level of consumption and associated level of leisure. Intuitively, a small φ renders the disutility from working inelastic and a large χ implies that the marginal product of labor *increases* with labor input. Conditional on the capital stock, different combinations of consumption and leisure thus satisfy the intratemporal first-order condition.

Figure 6.5 illustrates the economy's response to a sunspot shock at date $t = 3$ under the assumption that no additional sunspot shocks occur later in time that is, the variables assume their expected values from date $t = 4$ onwards. The simulation assumes logarithmic utility of consumption and very inelastic disutility of labor ($\varphi \approx 0$); the intratemporal first-order condition therefore implies $\hat{c}_t(\epsilon^t) = \hat{w}_t(\epsilon^t)$.

The sunspot shock increases labor supply and the wage, consumption, investment and thus, next period's capital stock. Labor supply remains elevated for several periods and the interest rate therefore increases as well in spite of the higher capital stock. As a consequence, normalized consumption continues to rise for a while. Moreover, it remains elevated long after labor supply has reverted back to and below its steady-state level because of the additional output from the increased capital stock.

6.4 Bibliographic Notes

Becker (1965) analyzes labor supply and Lucas and Rapping (1969) study a model of intertemporal labor supply. Hansen (1985) and Rogerson (1988) study economies with

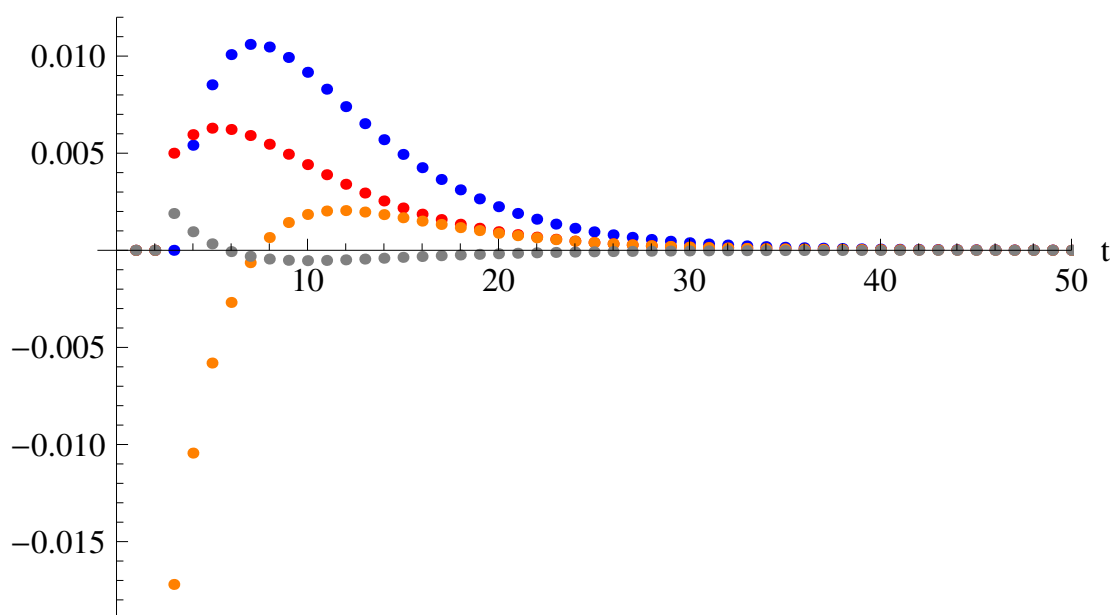


Figure 6.5: Effects of a sunspot shock when $\varphi \approx 0$. \hat{k}_t , $\hat{c}_t (= \hat{w}_t)$, $\hat{x}_t/20$, and \hat{R}_t are indicated in blue, red, orange and gray, respectively.

indivisible labor. The restrictions on technology in the model of subsection 6.2.1 are due to Uzawa (1961); the proof follows Schlicht (2006); and King, Plosser and Rebelo (1988) derive the restrictions on preferences, see also King, Plosser and Rebelo (2002, p. 94). The two-sector model in subsection 6.2.2 is due to Rebelo (1991) and the model with externalities follows Romer (1986). Barro and Sala-i-Martin (1995) and Acemoglu (2009) cover economic growth. Brock and Mirman (1972) analyze the stochastic growth model without labor-leisure choice. The real business cycle model is due to Kydland and Prescott (1982), Long and Plosser (1983), and King et al. (1988), see also Cooley (1995). King et al. (2002) carefully describe the solution strategy adopted here. The model with sunspots is due to Benhabib and Farmer (1994).

Beyond the material covered in the chapter, Krusell and Smith (1998) analyze an environment with idiosyncratic and aggregate risk; the state includes the complete wealth distribution. Krusell and Smith (1998) compute approximate equilibrium dynamics by representing the wealth distribution in terms of a few moments. To assess the cost of business cycles Lucas (1987, 3) compares the utility from two consumption sequences, one following a deterministic growth path and the other fluctuating around such a path. Alvarez and Jermann (2004) relate the marginal cost of consumption fluctuations to asset prices.

Chapter 7

The Open Economy

In the open economy, agents import and export goods and they save, borrow, and insure internationally. As a consequence, domestic production exceeds absorption by the *trade balance* and the economy accumulates *net foreign assets*. A positive trade balance implies an equal-sized increase in net foreign assets as it forces foreigners to borrow from domestic agents. Absent capital gains or losses, the *current account*—the trade balance plus income from net foreign assets—therefore equals the change in net foreign assets.

We study the determinants of net foreign assets, the trade balance, and the current account and analyze the welfare gains from intertemporal trade. Moreover, we examine factors that influence the real exchange rate as well as the consequences of international risk sharing.

7.1 Current Account and Net Foreign Assets

Consider the economy with homogeneous households, firms, and capital accumulation analyzed in section 3.1. Unlike in section 3.1 we assume that the economy is open such that the trade balance, tb_t , and the net foreign asset position, nfa_t , generally differ from zero. Letting a_t denote household assets and k_t the domestic capital stock, the trade balance and net foreign assets are given by

$$\begin{aligned}tb_t &= f(k_t, 1) - c_t - (k_{t+1} - k_t(1 - \delta)), \\nfa_t &= a_t - k_t,\end{aligned}$$

where f and c_t denote the standard neoclassical production function and consumption, respectively.

We assume that the economy is “small” that is, domestic saving has not affect on world interest rates. Since capital can freely move in and out of the country the domestic and international rental rate on capital, r_t , are identical. Firms take this rate as given and optimally choose labor and capital inputs. The firms’ optimality conditions

combined with the labor market clearing condition imply

$$\begin{aligned} f_K(k_t, 1) &= r_t, \\ f_L(k_t, 1) &= w_t. \end{aligned}$$

The first condition pins down the capital stock and the second determines the wage, w_t . Note that with constant world rental rates and time invariant technology the domestic capital stock and the wage are constant at all times. The domestic capital stock and consumption are decoupled because the economy borrows or lends in world capital markets.

The remaining equilibrium conditions include the dynamic budget constraint of households and the Euler equation,

$$\begin{aligned} a_{t+1} &= a_t R_t + w_t - c_t, \\ u'(c_t) &= \beta R_{t+1} u'(c_{t+1}), \end{aligned}$$

respectively, where the gross interest rate reflects the rental rate and depreciation, $R_t = 1 + r_t - \delta$. In addition, households satisfy a transversality condition. Note that the current account, ca_t , satisfies

$$\begin{aligned} ca_t &\equiv tb_t + (R_t - 1)nfa_t = f(k_t, 1) - c_t - (k_{t+1} - k_t(1 - \delta)) + (R_t - 1)nfa_t \\ &= f(k_t, 1) - c_t - k_{t+1} + k_t(1 - \delta) + (nfa_{t+1} + k_{t+1} - k_t R_t - w_t + c_t - nfa_t) \\ &= f(k_t, 1) - w_t - k_t(R_t - 1 + \delta) + nfa_{t+1} - nfa_t = nfa_{t+1} - nfa_t, \end{aligned}$$

where we use the definition of net foreign assets as well as the budget constraints of households and firms.

To solve for equilibrium consumption we iterate the dynamic budget constraint forward (see subsections 2.1.2–2.1.3). Assuming for simplicity that $R_t = R = \beta^{-1}$ we find

$$c_t = \rho nfa_t R + \rho \sum_{s=0}^{\infty} R^{-s} w_{t+s}, \quad (7.1)$$

where we define the annuity factor $\rho \equiv (\sum_{s=0}^{\infty} R^{-s})^{-1} = (R - 1)/R$. Intuitively, equilibrium consumption reflects lifetime wealth exactly as in the partial equilibrium model studied in chapter 2. Unlike in closed-economy general equilibrium models, however, the interest rate is exogenous and (under our assumption about R) equilibrium consumption is constant because the open capital account allows for domestic saving and investment and thus, consumption and capital accumulation to be decoupled.

Now abstract from capital, $k_t = 0$, and let $\{w_{t+s}\}_{s \geq 0}$ in equation (7.1) denote a, potentially time varying endowment sequence. The trade balance then equals $tb_t = w_t - c_t$ and the current account is given by

$$ca_t \equiv tb_t + (R - 1)nfa_t = w_t - c_t + nfa_{t+1} - w_t + c_t - nfa_t = nfa_{t+1} - nfa_t,$$

where we use the household's budget constraint. Let wp_t denote the *permanent income* or annuity corresponding to the endowment sequence:

$$wp_t \sum_{s=0}^{\infty} R^{-s} \equiv \sum_{s=0}^{\infty} R^{-s} w_{t+s} \Leftrightarrow wp_t \equiv \rho \sum_{s=0}^{\infty} R^{-s} w_{t+s}.$$

From the household's dynamic budget constraint, consumption equals $nfa_t R + w_t - nfa_{t+1}$, and from optimality condition (7.1), it equals $\rho nfa_t R + wp_t$. Equalizing the two expressions implies $nfa_t R(\rho - 1) + nfa_{t+1} = w_t - wp_t$ or

$$ca_t = nfa_{t+1} - nfa_t = w_t - wp_t. \quad (7.2)$$

Condition (7.2) states that net foreign assets increase (decrease) when income exceeds (falls short of) permanent income. This reflects the same consumption smoothing motive as the savings choice of a household in partial equilibrium (see chapter 2).

Introducing endowment risk does not affect these findings (except that w_{t+s} in condition (7.1) is replaced by $\mathbb{E}_t[w_{t+s}]$) as long as certainty equivalence holds (see subsection 4.1.1).

7.2 Real Exchange Rate

Suppose now that the endowment has two components, a non-tradable component, w_t^N , and a tradable component, w_t^T . Non-tradables only can be consumed domestically (think of haircuts or similar services) while tradables can be consumed abroad and costlessly shipped. The tradable good serves as numeraire and the price of non-tradables is denoted p_t .

Household consumption is an aggregate of the tradable and the non-tradable good,

$$c_t = c(c_t^T, c_t^N). \quad (7.3)$$

Its price, which increases in p_t , is denoted \mathcal{P}_t (see subsection 2.2.3 for the formula of \mathcal{P}_t). The *real exchange rate* is the price of one unit of domestic consumption relative to the price of a consumption unit abroad which we normalize to unity. The real exchange rate thus equals \mathcal{P}_t and it increases in p_t —a rising price of non-tradables implies a real exchange rate appreciation.

The household takes world interest rates as given and maximizes $\sum_{t=0}^{\infty} \beta^t u(c_t)$ subject to (7.3), the dynamic budget constraint,

$$\begin{aligned} a_{t+1} &= a_t R_t + w_t^T + w_t^N p_t - c_t \mathcal{P}_t \\ &= a_t R_t + w_t^T + w_t^N p_t - c_t^T - c_t^N p_t \end{aligned}$$

(assets are denominated in the numeraire), and a no-Ponzi-game condition. The first-order conditions are given by (see subsection 2.2.3)

$$u'(c_t)/\mathcal{P}_t = \beta R_{t+1} u'(c_{t+1})/\mathcal{P}_{t+1}, \quad (7.4)$$

$$u'(c_t) c_T(c_t^T, c_t^N) = \beta R_{t+1} u'(c_{t+1}) c_T(c_{t+1}^T, c_{t+1}^N), \quad (7.5)$$

$$p_t = \frac{c_N(c_t^T, c_t^N)}{c_T(c_t^T, c_t^N)}, \quad (7.6)$$

where $c_N(c_t^T, c_t^N)$ and $c_T(c_t^T, c_t^N)$ denote partial derivatives. Condition (7.4) represents the Euler equation for c_t ; note that the own rate of interest for the consumption index equals $R_{t+1}\mathcal{P}_t/\mathcal{P}_{t+1}$. Condition (7.5) gives the Euler equation for tradable consumption whose own rate of interest equals R_{t+1} . Finally, condition (7.6) equalizes the price of non-tradable consumption in terms of tradable consumption and the corresponding marginal rate of substitution.

Assume that $R_t = \beta^{-1}$ and $w_t^N = w^N$ in all periods. Imposing market clearing, $c_t^N = w^N$, the second and third optimality condition then reduce to

$$\begin{aligned} u'(c_t)c_T(c_t^T, w^N) &= u'(c_{t+1})c_T(c_{t+1}^T, w^N), \\ p_t &= \frac{c_N(c_t^T, w^N)}{c_T(c_t^T, w^N)}. \end{aligned}$$

From the first condition, the consumption index and both its components are constant over time. Domestic market clearing ($c_t^N = w^N$) and the household's intertemporal budget constraint then imply that c_t^T equals the annuity value of net foreign assets plus permanent income from the tradable endowment sequence (condition (7.1) with c_t replaced by c_t^T and w_{t+s} replaced by w_{t+s}^T). Tradable consumption thus increases in the initial net asset position and the permanent income from tradable goods. In a slightly extended model with differentiated export and import goods, it also increases in the *terms of trade*—the price of exports relative to imports—since improved terms of trade effectively increase the market value of the tradable endowment.

The second condition pins down the real exchange rate. Recall that non-tradable consumption is fixed (by domestic market clearing) while tradable consumption reflects the net asset position and the tradable endowment sequence (possibly accounting for the terms of trade). Higher net foreign assets or higher permanent income from tradables therefore increase the marginal rate of substitution on the right-hand side of the condition and thus, the price of non-tradables and the real exchange rate. Intuitively, higher household wealth raises the demand for tradables and non-tradables but with the latter in fixed supply, their equilibrium price must rise for markets to clear. We conclude that a wealthier economy (measured in terms of tradables) or one with a stronger preference for non-tradables has a more appreciated real exchange rate.

Over longer horizons, factors of production can be reallocated between sectors, implying that tradable and non-tradable output no longer are exogenous. This undermines the link between household wealth and the real exchange rate; in fact, the latter may be completely determined on the supply side.

To see this, relax the endowment assumption and suppose that competitive domestic firms employ capital and labor to produce goods. Output of tradables and non-tradables, respectively, is given by $A_t^T f^T(K_t^T, L_t^T)$ and $A_t^N f^N(K_t^N, L_t^N)$ where A_t^T and A_t^N denote productivity levels and the arguments of the constant returns to scale functions f^T and f^N denote capital and labor inputs in the two sectors. Capital is internationally mobile and earns the exogenous rental rate r_t while labor is mobile across sectors and earns the wage w_t .

For competitive firms to produce both goods the marginal value products of all

inputs must equal their respective rental rates, implying

$$\begin{aligned} A_t^T f_K^T(K_t^T, L_t^T) &= r_t, \\ p_t A_t^N f_K^N(K_t^N, L_t^N) &= r_t, \\ A_t^T f_L^T(K_t^T, L_t^T) &= w_t, \\ p_t A_t^N f_L^N(K_t^N, L_t^N) &= w_t. \end{aligned}$$

Due to constant returns to scale, the marginal products are functions of the respective capital-labor ratios, k_t^T or k_t^N . Conditional on A_t^T, A_t^N, r_t (and independently of household preferences), the four conditions thus pin down k_t^T, k_t^N, w_t , and p_t .

Higher tradable-sector productivity raises the price of non-tradables. This follows from the fact that a higher A_t^T raises k_t^T (from the first condition) and thus, w_t (from the third condition); that a higher w_t raises either p_t or k_t^N (from the fourth condition); and that p_t and k_t^N adjust in the same direction (from the second condition, because A_t^N and r_t are fixed). Intuitively, higher productivity in the tradable sector at given rental rates increases equilibrium wages. To attract workers non-tradable sector firms must pay higher wages even if their productivity is unchanged. For the marginal value products of capital and labor in the non-tradable sector to remain unchanged and rise, respectively, both p_t and k_t^N must rise. Similar reasoning establishes that an increase in non-tradable-sector productivity lowers p_t .

For an alternative perspective, consider the zero-profit conditions of firms,

$$\begin{aligned} A_t^T f^T(k_t^T, 1) &= w_t + k_t^T r_t, \\ p_t A_t^N f^N(k_t^N, 1) &= w_t + k_t^N r_t, \end{aligned}$$

which are implied by constant returns to scale and competition. Totally differentiating the zero-profit condition in the tradable sector (holding the rental rate fixed) and using the first-order condition with respect to K_t^T to cancel terms yields $dA_t^T f^T(k_t^T, 1) = dw_t$. This can be expressed as $\hat{A}_t^T = \hat{w}_t \sigma_t^T$ where a circumflex denotes an infinitesimal relative deviation and $\sigma_t^T \equiv w_t / (A_t^T f^T(k_t^T, 1))$ denotes the labor share in the tradable sector. Similarly, totally differentiating the zero-profit condition in the non-tradable sector, using the first-order condition with respect to K_t^N to cancel terms and letting $\sigma_t^N \equiv w_t / (p_t A_t^N f^N(k_t^N, 1))$ yields $\hat{p}_t + \hat{A}_t^N = \hat{w}_t \sigma_t^N$.

Combining the two expressions, we find

$$\hat{p}_t = \hat{A}_t^T \frac{\sigma_t^N}{\sigma_t^T} - \hat{A}_t^N,$$

which confirms that an increase in tradable-sector productivity raises p_t while an increase in non-tradable-sector productivity decreases it. Intuitively, an increase in tradable-sector productivity lowers the unit cost of the tradable good. As a consequence, competition pushes up wages and the non-tradable sector avoids losses only if its output price rises. If productivity in both sectors rises, p_t still increases as long as tradable-sector productivity grows more quickly and the labor share in the non-tradable sector is larger than in the tradable sector. If the two conditions are satisfied, the model

thus explains the *Baumol-Bowen* effect—the secular increase of the relative price of non-tradables—as well as the *Harrod-Balassa-Samuelson* effect, namely real appreciations in countries with faster productivity growth and thus, higher incomes.

A change in the world interest rate affects the real exchange rate too. Similar calculations to the previous ones show that, for constant productivity levels,

$$\hat{p}_t = \frac{dr_t}{p_t f^N} (k_t^N - k_t^T) = \frac{\hat{r}_t}{\sigma_t^T} (\sigma_t^T - \sigma_t^N).$$

A higher rental rate thus causes a real depreciation as long as the capital-labor ratio in the tradable sector is higher than in the non-tradable sector. This result mirrors the *Stolper-Samuelson theorem* according to which a price change benefits the production factor that is employed more intensively in the expanding sector.

7.3 Gains From Trade

Trade allows countries to mutually exploit *comparative advantages* that result from relative productivity or endowment differences. In addition to static gains, opening economies up generates gains from intertemporal trade (saving and borrowing) and from risk sharing.

Focusing first on the gains from intertemporal trade, consider a two-period setting. The economy is endowed with an initial stock of capital, a_0 ; one unit of labor in each period; and a constant returns to scale production function, f , that is the same as in the rest of the world. Domestic assets equal the domestic capital stock plus net foreign assets, $a_t = k_t + nfa_t$. The budget constraints of the economy are given by

$$\begin{aligned} c_0 &= f(k_0, 1) - (k_0 - a_0)r_0 + a_0(1 - \delta) - a_1, \\ c_1 &= f(k_1, 1) - (k_1 - a_1)r_1 + a_1(1 - \delta). \end{aligned}$$

When the economy is closed, $a_t = k_t$. After opening up, capital in- or outflows assure that the domestic and international rental rate on capital, r_t , are identical and the domestic (and international) capital-labor-ratio satisfies

$$f_K(k_t, 1) = r_t.$$

Suppose first that households are homogenous. The representative agent maximizes $u(c_0) + \beta u(c_1)$ and chooses a_1 . When the economy is closed this choice satisfies the Euler equation $u'(c_0) = \beta(f_K(a_1, 1) + 1 - \delta)u'(c_1)$. When it is open, in contrast, capital flows in or out, $k_t \neq a_t$, and the choice of a_1 satisfies the Euler equation $u'(c_0) = \beta(r_1 + 1 - \delta)u'(c_1)$. Using the budget constraints, a first-order approximation of the welfare effect from capital account liberalization about the closed-economy capital stocks, a_0 and a_1 , yields

$$u'(c_0)(f_K(a_0, 1) - r_0)(k_0 - a_0) + \beta u'(c_1)(f_K(a_1, 1) - r_1)(k_1 - a_1).$$

(The indirect welfare effect of the induced change in a_1 equals zero, due to the envelope condition.)

Note that in each period, the product $(f_K(a_t, 1) - r_t)(k_t - a_t)$ is positive, implying that the disposable income in each period rises and the welfare effect is positive. Intuitively, when capital flows into the economy, $k_t > a_t$, the marginal product of capital falls from $f_K(a_t, 1)$ to r_t and output benefits from productive, cheap inframarginal units of foreign capital. Conversely, when capital flows out, $k_t < a_t$, the marginal product rises and domestic production falls but the inframarginal units of freed capital earn a rental rate abroad that exceeds the marginal product in autarky.

While capital in- or outflows unambiguously raise disposable income they affect the returns on capital and labor unequally. For example, if the international capital-labor ratio exceeds the domestic ratio in autarky then opening up the capital account lowers the rental rate but raises domestic wages. With homogeneous households this change of factor incomes is of no importance since all income goes to the same representative household. With heterogeneous households, in contrast, unequal effects on factor incomes imply that capital account liberalization may benefit some groups while hurting others.

To see this, consider one change to the setup studied above: The initial capital stock now is owned by an old generation that dies after the first period. The change of rental rate in the first period then exclusively affects the old generation while the wage change and the change of second-period rental rate only affects the young generation. Since the rental rate in the first period moves opposite to the capital-labor ratio we can immediately conclude that capital inflows after a capital account liberalization harm the initial old generation although the inflows increase aggregate disposable income. With sufficiently high transfers from the young to the old generation, however, all cohorts benefit.

The (aggregate) gains from intertemporal trade reflect efficiency gains due to the international equalization of marginal rates of transformation and substitution. Similar gains arise from the equalization of the marginal rate of substitution across histories that is, from international risk sharing.

7.4 International Risk Sharing

Consider the framework with tradable and non-tradable endowments analyzed in section 7.2 and suppose that endowments are risky and households can trade a complete set of Arrow-Debreu securities denominated in the tradable good (the numeraire). Assuming identical psychological discount factors, the equilibrium marginal rate of substitution between consumption of the tradable good at date t , history ϵ^t , and at date $t + s$, history ϵ^{t+s} ,

$$\frac{u'(c_{t+s}(\epsilon^{t+s}))}{u'(c_t(\epsilon^t))} \frac{c_T(c_{t+s}^T(\epsilon^{t+s}), w_{t+s}^N(\epsilon^{t+s}))}{c_T(c_t^T(\epsilon^t), w_t^N(\epsilon^t))},$$

then is the same for all households in all countries. Equivalently, the marginal rate of substitution between the consumption index at the two dates and histories, corrected

for variation in the real exchange rate,

$$\frac{u'(c_{t+s}(\epsilon^{t+s}))/\mathcal{P}_{t+s}(\epsilon^{t+s})}{u'(c_t(\epsilon^t))/\mathcal{P}_t(\epsilon^t)},$$

also is the same across countries. This follows from the risk-sharing result in section 4.2 (see condition (4.2)) and the equilibrium conditions (7.4) and (7.5) once we allow for risk. With complete markets, marginal utility of the consumption index thus is not perfectly correlated internationally unless the real exchange rate is constant. Marginal utility grows faster in countries whose consumer price index grows more slowly.

With incomplete markets, the correlation is even weaker. For example, when only a risk-free bond (denominated in tradables) is traded then the expected marginal rate of substitution corrected for the price index,

$$\frac{\mathbb{E}_t[u'(c_{t+s}(\epsilon^{t+s}))/\mathcal{P}_{t+s}(\epsilon^{t+s})]}{u'(c_t(\epsilon^t))/\mathcal{P}_t(\epsilon^t)},$$

is equalized across countries. This follows from the stochastic Euler equation variant of condition (7.4).

7.5 Bibliographic Notes

Buiter (1981), Obstfeld (1982), Sachs (1981), and Svensson and Razin (1983) contain early models of the current account with optimizing models agents. The Harrod-Balassa-Samuelson effect is named after Harrod (1933), Balassa (1964), and Samuelson (1964). Dornbusch, Fischer and Samuelson (1977) propose a tractable model of the terms of trade. Samuelson (1939) discusses the (static) gains from trade; and Fried (1980) or Buiter (1981) analyze intergenerational welfare effects of capital account liberalization. Backus and Smith (1993) analyze international risk sharing with non-tradable goods.

Chapter 8

Real Frictions

8.1 Capital Adjustment Costs

In the benchmark model, investment contributes one-for-one to the buildup of capital. We now relax this assumption and posit that the buildup of capital is subject to adjustment costs. As a consequence, firms solve dynamic optimization problems. Rather than renting capital on spot markets, they install it with a view on current and future adjustment costs.

8.1.1 Convex Adjustment Costs and Tobin's q

Suppose that a buildup of capital equal to I_t requires resources $I_t + A(I_t, K_t)$ where A denotes an adjustment cost function. For now, we abstract from depreciation ($\delta = 0$) and assume the following properties of A : Adjustment costs are weakly positive; equal to zero when there is no adjustment; as well as smooth and strictly convex. Formally, $A(I, K) \geq 0$; $A(0, K) = 0$; $A_I(0, K) = 0$; and $A_{II}(I, K) > 0$. An example of A that we will use is the function $A(I, K) = zI^2 / (2K)$ for some constant $z > 0$. Note that with this adjustment cost function, a larger preexisting capital stock reduces the adjustment cost per unit of capital buildup.

Consider a firm operating a neoclassical production function, f , that faces wages, w_t , and a constant (for simplicity) gross interest rate, R , and that chooses investment, I_t , and labor demand, L_t , to maximize profits. The Lagrangian associated with the firm's program reads

$$\mathcal{L} = \sum_{t=0}^{\infty} R^{-t} [f(K_t, L_t) - I_t - w_t L_t - A(I_t, K_t) - q_t (K_{t+1} - K_t - I_t)],$$

where the multiplier q_t is associated with the law of motion for installed capital. This multiplier—*Tobin's q* —represents the shadow value of installed capital relative to the price of “outside capital” or (investment) goods, which is normalized to unity. The

first-order conditions with respect to L_t , K_{t+1} , and I_t , respectively, are given by

$$\begin{aligned} f_L(K_t, L_t) &= w_t, \\ f_K(K_{t+1}, L_{t+1}) &= A_K(I_{t+1}, K_{t+1}) - q_{t+1} + Rq_t, \\ q_t &= 1 + A_I(I_t, K_t). \end{aligned}$$

The first condition represents the usual labor demand relation: Conditional on installed capital the firm equalizes the marginal product of labor and the wage. The second condition can be written as an asset pricing relation,

$$q_t = \frac{f_K(K_{t+1}, L_{t+1}) - A_K(I_{t+1}, K_{t+1}) + q_{t+1}}{R}.$$

It states that the shadow price of installed capital equals the discounted shadow price in the subsequent period, plus the discounted dividend from installed capital; the dividend in turn equals the marginal product of capital (as usual), net of the reduction in future adjustment costs due to the higher capital stock. Iterating the equation forward yields (absent bubbles)

$$q_0 = \sum_{t=1}^{\infty} \frac{f_K(K_t, L_t) - A_K(I_t, K_t)}{R^t}.$$

Note that in the absence of adjustment costs, $q_t = 1$ and $f_K(K_t, L_t) = R - 1$, corresponding to our findings in the baseline model.

According to the last condition, the shadow price of installed capital equals the replacement cost of capital plus the marginal adjustment cost. Due to the convexity of A , this condition yields a unique mapping from q_t to I_t (conditional on K_t)—for a given stock of installed capital, q is a sufficient statistic for investment.

The value of the firm at date $t = 0$, V_0 , is given by

$$V_0 = \sum_{t=1}^{\infty} \frac{f(K_t, L_t) - w_t L_t - A(I_t, K_t) - I_t}{R^t},$$

evaluated at the optimal investment and labor demand. If both f and A exhibit CRTS, then the value of a marginal unit of installed capital, q_t , and the average value of installed capital, V_t/K_{t+1} , coincide (“marginal and average q coincide”). This follows from

$$\begin{aligned} K_{t+1}q_t &= \frac{f_K(K_{t+1}, L_{t+1})K_{t+1} - A_K(I_{t+1}, K_{t+1})K_{t+1} + q_{t+1}K_{t+1}}{R} \\ &= \frac{f(K_{t+1}, L_{t+1}) - w_{t+1}L_{t+1} - A_K(I_{t+1}, K_{t+1})K_{t+1} + (1 + A_I(I_{t+1}, K_{t+1}))(K_{t+2} - I_{t+1})}{R} \\ &= \frac{f(K_{t+1}, L_{t+1}) - w_{t+1}L_{t+1} - A(I_{t+1}, K_{t+1}) - I_{t+1} + q_{t+1}K_{t+2}}{R} \\ &= \sum_{s=t+1}^{\infty} \frac{f(K_s, L_s) - w_s L_s - A(I_s, K_s) - I_s}{R^{s-t}} = V_t. \end{aligned}$$

The fact that marginal and average q coincide implies that the gross rate of return on shares of the firm equals R .

Consider the quadratic adjustment cost function introduced above, let $r \equiv R - 1$, and assume for simplicity that labor demand is fixed at L . The optimality conditions then read

$$\begin{aligned} I_t z &= (q_t - 1)K_t, \\ Rq_t - q_{t+1} &= \\ f_K(K_{t+1}, L) + \frac{z}{2} \left(\frac{I_{t+1}}{K_{t+1}} \right)^2 &= f_K \left(K_t \left(1 + \frac{q_t - 1}{z} \right), L \right) + \frac{1}{2z} (q_{t+1} - 1)^2. \end{aligned}$$

The first equation states that the investment-capital ratio is proportional to $q_t - 1$. The second condition relates the marginal product of capital to q_t and q_{t+1} .

In steady state, the capital stock is constant at value K satisfying $r = f_K(K, L)$. Accordingly, the steady-state shadow price satisfies $q = 1$. A first-order Taylor expansion about the steady state yields the following linear dynamic system in the variables $K_t - K$ and $q_t - 1$:

$$\begin{aligned} K_{t+1} - K_t &= \frac{K}{z} (q_t - 1), \\ q_{t+1} - q_t &= \left(r - \frac{K f_{KK}(K, L)}{z} \right) (q_t - 1) - f_{KK}(K, L) (K_t - K). \end{aligned}$$

The matrix governing the system dynamics,

$$M \equiv \begin{bmatrix} 1 & K/z \\ -f_{KK}(K, L) & 1 + r - K f_{KK}(K, L)/z \end{bmatrix},$$

has one stable and one unstable eigenvalue. Since capital is predetermined while the shadow price is a “jump variable,” the linear system is saddle path stable: For any initial level of installed capital, K_0 , there exists a unique shadow price, q_0 , such that starting from (K_0, q_0) , the dynamic system prescribes a path that converges to the steady state.

Figure 8.1 illustrates the dynamics by means of a phase diagram in (K, q) space. The horizontal solid line depicts points at which the capital stock is constant, $K_{t+1} = K_t$. Below (above) that locus, $q_t < (>) 1$ and the capital stock falls (grows). The decreasing solid line depicts points with time invariant shadow prices, $q_{t+1} = q_t$. To the left (right) of that locus, the marginal product of capital is high (low) and the shadow price falls (rises). The dotted paths indicate adjustment paths starting from two initial capital stocks, K^{low} and K^{high} . All paths satisfy the two conditions governing system dynamics; only the blue paths follow the saddle path and converge to the steady state, $(K^{\text{ss}}, 1)$.

Using the phase diagram, we can analyze the effect of shocks on investment. Suppose for example that the firm is in steady state when it learns that productivity has

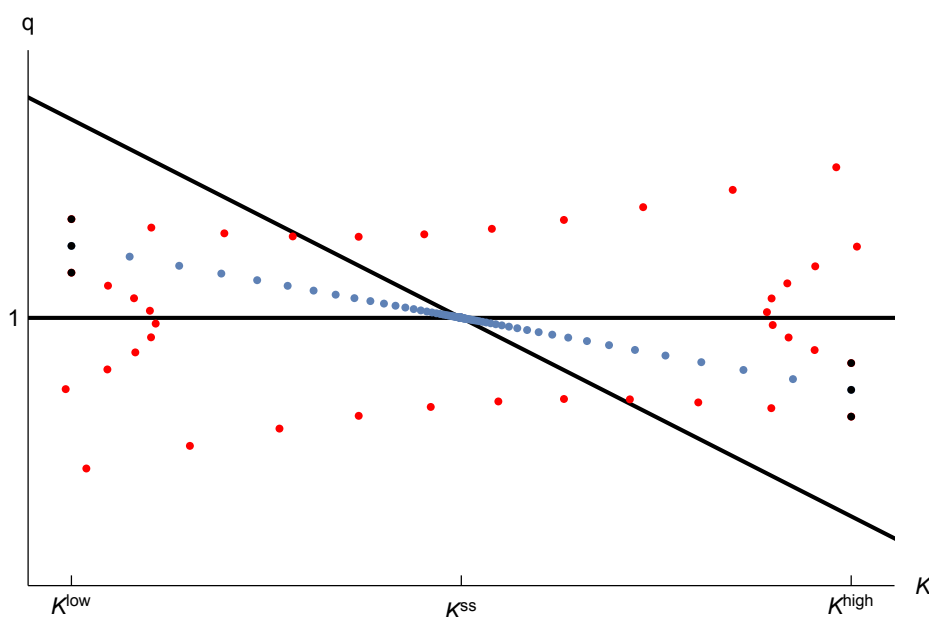


Figure 8.1: Dynamics of installed capital and its shadow price: Steady-state capital accumulation condition and shadow price relation (solid, in black) as well as dynamic adjustment paths off (red) and on (blue) the saddle path.

permanently increased (or the interest rate decreased). (Ex ante, the firm attached negligible probability to this event.) In the phase diagram in figure 8.1, this shock is reflected in an outward shift of the steady-state shadow price relation (the decreasing solid line) because for each level of capital the marginal product has increased. The saddle path therefore shifts out as well. While the installed capital cannot change on impact the shadow price and the value of the firm immediately rise. Intuitively, the value of installed capital increases because the marginal product is higher, and the marginal product is higher because it is costly to quickly adjust the capital stock. Over time, the firm builds up capital until the shadow price returns to its steady-state value of one.

In general equilibrium, the capital stock and q interact with household consumption and saving. As usual, the household Euler equation relates the growth rate of consumption to the gross interest rate which now equals

$$R_{t+1} = \frac{f_K(K_{t+1}, L_{t+1}) - A_K(I_{t+1}, K_{t+1}) + q_{t+1}}{q_t}.$$

The Euler equation augments the dynamic equilibrium conditions discussed above.

8.1.2 Non-Convex Adjustment Costs

If the adjustment cost function is not convex then q_t ceases to be a sufficient statistic for investment. We analyze this case in a two-period model and compare it to the situation

without adjustment costs or with convex adjustment costs.

Suppose that the firm has an initial stock of installed capital, K_0 ; chooses investment or disinvestment, I_0 ; and maximizes firm value, $[f(K_1, L) + K_1]/R - A(I_0, K_0) - I_0$, subject to the law of motion $K_1 = K_0 + I_0$. Consider first the case without adjustment costs, $A(I_0, K_0) \equiv 0$. Let K_1^* denote the optimal capital stock at date $t = 1$ in this case, that is $f_K(K_1^*, L) = r$. Clearly, in equilibrium, $I_0 = K_1^* - K_0$ and $q_0 = 1$.

Consider next the convex adjustment costs discussed above (and let $q_1 = 1$ and $A_K(I_1, K_1) = 0$). Equilibrium then is characterized by the conditions

$$\begin{aligned} q_0 &= \frac{f_K(K_1, L) + 1}{R}, \\ q_0 &= 1 + A_I(I_0, K_0). \end{aligned}$$

Investment or disinvestment is smaller than in the frictionless case. Moreover, unless $K_1^* = K_0$, $q_0 \neq 1$. The dotted and solid black lines in figure 8.2 illustrate the relation between K_0 , q_0 , and I_0 in the frictionless and the convex adjustment cost case, respectively.

As an example of non-convex adjustment costs, consider a fixed cost. Let $A(I_0, K_0) = A > 0$ iff $I \neq 0$ and zero otherwise. The optimal policy then consists of either not adjusting the capital stock at all, or fully adjusting it to the frictionless level. In the former case, q_0 differs from unity while in the latter, it does not. Either way, q_0 is not a sufficient statistic for investment. The blue lines in figure 8.2 illustrate the case with a fixed adjustment cost.

As another example of non-convex adjustment costs, consider proportional costs of adjustment, $A(I_0, K_0) = a|I_0|$, $a > 0$. Now, the firm faces a constant marginal cost of adjusting. If $K_1^* - K_0$ is "small" in absolute value then the marginal gain of adjusting is smaller than a ; consequently, $I_0 = 0$ and q_0 differs from unity. If the absolute value of $K_1^* - K_0$ is "large," in contrast, then the firm adjusts to the point where the marginal gain of further adjustment equals the marginal cost. That is, $I_0 \neq 0$ in that range but the adjustment is incomplete and q_0 does not reach unity. The red lines in figure 8.2 illustrate this case.

8.1.3 Option Value of Waiting

Suppose a firm must decide whether to invest I in a project that pays a dividend stream from the subsequent period onward. The firm's gross discount rate equals $R \geq 1$ and the present discounted value (PDV) of the dividend stream, as of the date at which it pays out for the first time, equals D , which can take two values: Either $D = H > RI$, or $D = L$. At date $t = 0$, the probability that $D = H$ equals p . At date $t = 1$, uncertainty is resolved and the firm learns whether D is high or low. Once the investment is undertaken, it may be liquidated; this generates a liquidation value, $V \leq I$. (If $V = 0$, the investment is irreversible.)

To assess whether it is optimal for the firm to invest at date $t = 0$ it is not sufficient to check whether the PDV of this strategy is positive. Rather, we need to check whether

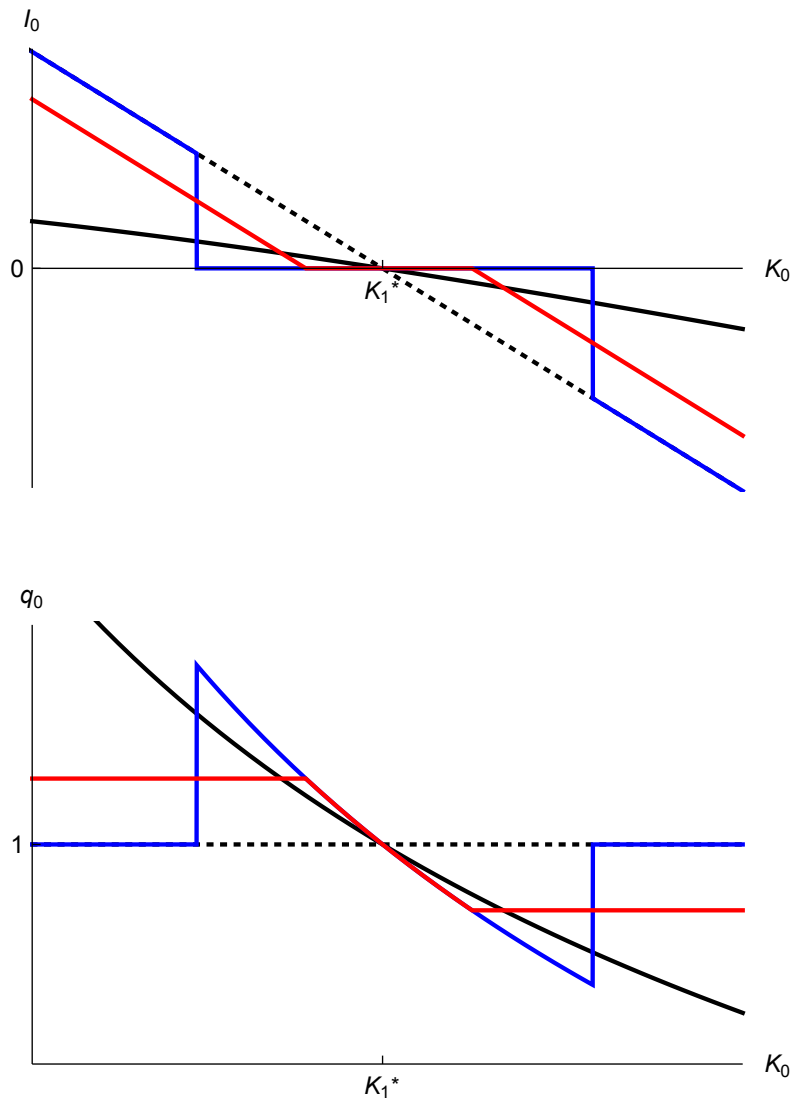


Figure 8.2: Convex and non-convex adjustment costs: Optimal investment (top) and q (bottom).

the PDV of investing at date $t = 0$ is higher than the PDV of not investing and *maintaining the option* to invest later. To derive the latter, note that the PDV at date $t = 1$ of having invested at date $t = 0$ equals $\max[V, D]$, while the PDV of not having invested equals $\max[0, -I + D/R]$. We conclude that it is optimal to invest at date $t = 0$ if

$$-I + \frac{pH + (1 - p) \max[V, L]}{R} \geq \frac{p(-I + H/R) + (1 - p) \max[0, -I + L/R]}{R}.$$

Otherwise, it is preferable at date $t = 0$ to “wait and see.”

Note that the preceding inequality differs from a naive investment rule that favors investment at date $t = 0$ whenever the left-hand side of the inequality is positive (possibly with $\max[V, L]$ replaced by L). The explanation for this difference lies in the fact that the *option to wait* is valuable when it is costly to undo the investment at a later stage in response to bad news.

To build intuition, suppose first that the investment always is profitable, $L/R \geq I$. The inequality above then reduces to

$$-I + \frac{pH + (1 - p)L}{R} \geq \frac{p(-I + H/R) + (1 - p)(-I + L/R)}{R},$$

which holds strictly when $R > 1$ and with equality otherwise. Intuitively, when the project always is profitable then waiting has a cost—it delays profits—but no benefit.

Suppose next that there is no discounting in the first two periods, $R = 1$, such that the inequality reduces to

$$-I + pH + (1 - p) \max[V, L] \geq p(-I + H) + (1 - p) \max[0, -I + L]$$

or, equivalently, $\max[V, L] \geq \max[I, L]$. This condition is violated whenever the low dividend stream is unprofitable ($L < I$) and the investment cannot fully be recouped through liquidation ($V < I$). Intuitively, when the investment can turn sour and there is no time cost of waiting, then waiting is preferable.

Suppose finally that the project is fully reversible, $V = I$. The inequality then reduces to

$$-I + \frac{pH + (1 - p) \max[I, L]}{R} \geq \frac{p(-I + H/R) + (1 - p) \max[0, -I + L/R]}{R}.$$

For $L \leq I$, this simplifies to the condition $pH/R \geq I$. When the project is fully reversible but the low dividend stream is unprofitable then waiting is preferable unless the high dividend stream alone is sufficient to recoup the investment outlay.

8.2 Labor Market Frictions

In the baseline model discussed in section 6.1, workers and firms meet on a competitive, frictionless labor market. The equilibrium wage is determined by the condition that labor demand equals supply. We now relax this assumption and introduce *search*

and *matching frictions*. Specifically, we assume that job seekers and firms trying to fill vacancies must invest resources to search for each other, and that they meet randomly. This affects both job creation and wage determination. When a job seeker and a firm with a vacancy meet they are in a bilateral monopoly rather than a competitive situation. Rather than waiting for another match, the job seeker is willing to accept any wage higher than the marginal rate of substitution, and the firm is willing to accept any wage lower than labor's marginal product; there is thus space for negotiation. Accordingly, we introduce a new mechanism to determine wages.

8.2.1 Economy

The economy is inhabited by a representative firm and a representative household with a continuum of household members that insure each other, as in the extensive margin model considered in subsection 6.1.1. At date t , a fraction x_t of household members consumes leisure; a fraction y_t supplies labor; and a fraction $z_t = 1 - x_t - y_t$ searches for a job. The firm accumulates capital, k_t , which it uses jointly with hired labor, $y_t(1 - v_t)$, to produce output subject to the CRTS production function, f . It employs the remaining labor force, $y_t v_t$, to recruit new workers for its vacancies. Productivity, A_t , is exogenous and labor augmenting.

Vacancies are filled, and job seekers are employed, according to a CRTS matching function, m . When the household has z_t job seekers and the firm tries to fill v_t vacancies then

$$m(z_t, v_t) = z_t m\left(1, \frac{v_t}{z_t}\right) \equiv z_t m(1, \theta_t)$$

vacancies are filled and job seekers get employed. Variable θ_t denotes *labor market tightness*, the ratio of vacancies to unemployed (job seekers). At date t , the share $\pi(\theta_t) \equiv m(1, \theta_t)/\theta_t = m(z_t, v_t)/v_t$ of vacancies is filled; the function π is decreasing. Correspondingly, the share $\theta_t \pi(\theta_t) = m(z_t, v_t)/z_t$ of job seekers (which increases in θ_t) gets employed. Both households and firms take labor market tightness as given. Employment is a state variable and employment relationships end with an exogenous probability, s .

For notational simplicity, we abstract from aggregate (productivity) risk and formulate the equilibrium conditions recursively. Let a_t denote household financial assets at date t ; and q_{t+1} the price at date t of an asset that pays off one unit at the subsequent date.

8.2.2 Firms

The firm's value function, W , satisfies

$$\begin{aligned} W(k_t, y_t, t) &= \max_{k_{t+1}, v_t} cf_t + q_{t+1}W(k_{t+1}, y_{t+1}, t+1) \\ \text{s.t.} \quad cf_t &= f(A_t, k_t, y_t(1 - v_t)) + k_t(1 - \delta) - k_{t+1} - y_t w_t, \\ y_{t+1} &= y_t(1 - s) + \pi(\theta_t)y_t v_t, \\ 0 &\leq v_t \leq 1, \end{aligned}$$

where cf_t denotes cash flow; $\delta \geq 0$ is the depreciation rate; and w_t denotes the wage. The first constraint defines cash flow as output net of gross investment and the wage bill; the second represents the law of motion for employment from the firm's perspective; and the third reflects the fact that the number of recruiters cannot exceed the number of employees. Note that $y_t v_t$ equals the number of the firm's vacancies.

The first-order conditions with respect to capital and recruiting, respectively, are

$$\begin{aligned} 1 &= q_{t+1}W_k(k_{t+1}, y_{t+1}, t+1), \\ f_y(A_t, k_t, y_t(1 - v_t)) &= \pi(\theta_t)q_{t+1}W_y(k_{t+1}, y_{t+1}, t+1). \end{aligned}$$

The firm equates the resource cost of investment to the discounted marginal increase in firm value due to a higher capital stock; and the output loss due to more intense recruiting to the discounted marginal increase in firm value from higher employment, weighted by the probability of filling a vacancy.

Using the envelope conditions for capital and employment,

$$\begin{aligned} W_k(k_t, y_t, t) &= f_k(A_t, k_t, y_t(1 - v_t)) + 1 - \delta, \\ W_y(k_t, y_t, t) &= f_y(A_t, k_t, y_t(1 - v_t)) - w_t + (1 - s)q_{t+1}W_y(k_{t+1}, y_{t+1}, t+1), \end{aligned}$$

we find

$$1 = q_{t+1}(f_k(A_{t+1}, k_{t+1}, y_{t+1}(1 - v_{t+1})) + 1 - \delta), \quad (8.1)$$

$$\begin{aligned} f_y(A_t, k_t, y_t(1 - v_t)) &= \pi(\theta_t)q_{t+1} \\ &\times \left(f_y(A_{t+1}, k_{t+1}, y_{t+1}(1 - v_{t+1})) \left(1 + \frac{1 - s}{\pi(\theta_{t+1})} \right) - w_{t+1} \right), \end{aligned} \quad (8.2)$$

$$W_y(k_t, y_t, t) = f_y(A_t, k_t, y_t(1 - v_t)) \left(1 + \frac{1 - s}{\pi(\theta_t)} \right) - w_t. \quad (8.3)$$

The first equation is standard. The second equates the cost and benefit of recruiting. The last condition states that the marginal value of employment equals the marginal product of labor, net of the wage, plus the recruitment cost that the firm saves when employment is higher. (One employee at date t generates $1 - s$ units of employment at date $t + 1$; $1/\pi(\theta_t)$ recruiters at date t increase employment at date $t + 1$ by one unit. An additional employee at date t thus saves $(1 - s)/\pi(\theta_t)$ recruiters who can instead be employed in production.)

8.2.3 Households

The date t period utility of the representative household is given by $u(c_t) - \gamma(y_t + z_t)$, where c_t denotes consumption and $\gamma > 0$ measures the disutility of work or job search. We assume logarithmic preferences. The household's value function, V , thus satisfies

$$\begin{aligned} V(a_t, y_t, t) &= \max_{a_{t+1}, z_t} \ln(c_t) - \gamma(y_t + z_t) + \beta V(a_{t+1}, y_{t+1}, t + 1) \\ \text{s.t.} \quad c_t &= a_t + w_t y_t - q_{t+1} a_{t+1}, \\ y_{t+1} &= y_t(1 - s) + \theta_t \pi(\theta_t) z_t, \\ 0 &\leq z_t \leq 1 - y_t. \end{aligned}$$

The household also satisfies a natural borrowing limit. The first constraint is the budget constraint. The second constraint represents the law of motion for employment from the household's perspective, and the third constraint is the time use constraint.

The first-order condition with respect to assets,

$$\frac{1}{c_t} q_{t+1} = \beta V_a(a_{t+1}, y_{t+1}, t + 1),$$

relates the intertemporal marginal rate of substitution to the price. The optimality condition with respect to labor market search,

$$\gamma = \beta \theta_t \pi(\theta_t) V_y(a_{t+1}, y_{t+1}, t + 1),$$

equates the cost of job search (foregone utility from leisure) and the marginal benefit; the latter equals the discounted continuation value of employment, multiplied by the probability of a match.

Using the envelope conditions,

$$\begin{aligned} V_a(a_t, y_t, t) &= \frac{1}{c_t}, \\ V_y(a_t, y_t, t) &= \frac{w_t}{c_t} - \gamma + \beta(1 - s) V_y(a_{t+1}, y_{t+1}, t + 1), \end{aligned}$$

we arrive at the equilibrium conditions

$$q_{t+1} = \beta \frac{c_t}{c_{t+1}}, \quad (8.4)$$

$$\gamma = \beta \theta_t \pi(\theta_t) \left(\frac{w_{t+1}}{c_{t+1}} - \gamma + \gamma \frac{1 - s}{\theta_{t+1} \pi(\theta_{t+1})} \right), \quad (8.5)$$

$$V_y(a_t, y_t, t) = \frac{w_t}{c_t} - \gamma + \gamma \frac{1 - s}{\theta_t \pi(\theta_t)}. \quad (8.6)$$

The first condition is standard. The second equates the cost and benefit of job search. The third condition states that higher employment generates two benefits: Marginal utility due to higher labor income, net of the disutility from work; and marginal utility from leisure because of reduced job search. (An additional unit of employment at date t generates $1 - s$ units at date $t + 1$, thus saving the household $(1 - s) / (\theta_t \pi(\theta_t))$ job seekers at date t .)

8.2.4 Market Clearing and Wage Determination

Goods market clearing implies

$$c_t = f(A_t, k_t, y_t(1 - v_t)) + k_t(1 - \delta) - k_{t+1}. \quad (8.7)$$

Employment, y_t , is a state variable that changes in response to exogenous separations and endogenous hires. When a job seeker and a firm with a vacancy meet the two parties negotiate; if they can agree on a wage then they form a new employment relationship.

Let $\tilde{V}_y(a_t, y_t, t, \Delta w_t)$ and $\tilde{W}_y(k_t, y_t, t, \Delta w_t)$ denote the value of the household and the firm, respectively, of a marginal hire whose wage at date t exceeds the equilibrium wage by Δw_t , but in the future equals the equilibrium wage. From the value functions derived earlier,

$$\begin{aligned} \tilde{V}_y(a_t, y_t, t, \Delta w_t) &= \frac{\Delta w_t}{c_t} + V_y(a_t, y_t, t), \\ \tilde{W}_y(k_t, y_t, t, \Delta w_t) &= -\Delta w_t + W_y(k_t, y_t, t). \end{aligned}$$

With *Nash bargaining*, the equilibrium wage maximizes the weighted geometric average of the joint surplus of the negotiating parties. The joint surplus of the household and the firm from forming an employment relationship is given by

$$\mathcal{J}(\Delta w_t) \equiv (\tilde{V}_y(a_t, y_t, t, \Delta w_t))^\phi (\tilde{W}_y(k_t, y_t, t, \Delta w_t))^{1-\phi},$$

where ϕ and $1 - \phi$ denote the bargaining weights of the job seeker and the firm, respectively.¹ In equilibrium, the wage thus solves $d\mathcal{J}(0)/d\Delta w_t = 0$ or equivalently,

$$\phi \frac{1}{c_t} \frac{1}{V_y(a_t, y_t, t)} = (1 - \phi) \frac{1}{W_y(k_t, y_t, t)}.$$

Using conditions (8.3) and (8.6) and solving for the wage, we find

$$\begin{aligned} w_t &= \gamma c_t \left(1 - \frac{1-s}{\theta_t \pi(\theta_t)} \right) \\ &\quad + \phi \left\{ f_y(A_t, k_t, y_t(1 - v_t)) \left(1 + \frac{1-s}{\pi(\theta_t)} \right) - \gamma c_t \left(1 - \frac{1-s}{\theta_t \pi(\theta_t)} \right) \right\}. \end{aligned} \quad (8.8)$$

The equilibrium wage has two components, represented by the two terms on the right-hand side of equation (8.8). First, the household's opportunity cost or outside value, namely the forgone utility from leisure (in consumption terms) net of the saved search cost. And second, the share ϕ of the joint surplus. The latter equals the sum of the household's and the firm's marginal values from a hire, net of the outside values of the two parties.

¹We have assumed that agents take the wage as given when choosing a or k although these state variables may affect the bargaining outcome. To reconcile these features one could assume that counter parties do not observe individual assets or that the firm negotiates a uniform wage for all workers.

8.2.5 Equilibrium

In equilibrium, household assets represent the value of the representative firm; labor market tightness reflects optimal firm and household choices, $\theta_t \equiv y_t v_t / z_t$; and the laws of motion for employment as perceived by the firm and the household reduce to

$$y_{t+1} = y_t(1 - s) + \pi \left(\frac{y_t v_t}{z_t} \right) y_t v_t. \quad (8.9)$$

Conditional on the state variables in the initial period an equilibrium is characterized by (8.1), (8.2), (8.4), (8.5), (8.7)–(8.9) as well as the definition of θ_t ; the household budget constraint; and the borrowing limit.

Using (8.4) and (8.8), we can re-express conditions (8.2) and (8.5) as

$$\frac{f_y(A_t, k_t, y_t(1 - v_t))}{c_t} = \beta \pi(\theta_t)(1 - \phi)\Omega_{t+1}, \quad (8.10)$$

$$\gamma = \beta \theta_t \pi(\theta_t) \phi \Omega_{t+1}, \quad (8.11)$$

respectively, where we define (dropping arguments of the marginal product function for legibility)

$$\Omega_{t+1} \equiv \frac{f_y(t+1)}{c_{t+1}} - \gamma + \frac{1-s}{\pi(\theta_{t+1})} \left(\frac{f_y(t+1)}{c_{t+1}} + \frac{\gamma}{\theta_{t+1}} \right).$$

We will use these conditions below.

8.2.6 Constrained Pareto Optimality

Job seekers and firms with vacancies are *rationed*: They are willing to form employment relationships at the going wage but are unable to do so until being matched. When labor market tightness is high then firms are typically rationed for an extended period while job seekers quickly find a job; when tightness is low the situation is reversed. In either case, job seekers and firms exert negative *congestion externalities* and positive *thick market externalities*: Their search renders it harder for agents of the same type, but easier for agents of the other type to be matched.

A social planner cannot avoid the matching friction. But the planner internalizes the search externalities, in contrast to firms and job seekers in the decentralized equilibrium. As a consequence, the (constrained optimal) social planner allocation typically differs from the equilibrium allocation. It is only when the bargaining weight of job seekers and firms assumes a particular value that the equilibrium allocation is constrained efficient.

To see this, consider the social planner's program. Letting P denote the value function of the social planner, we have

$$\begin{aligned} P(k_t, y_t, t) &= \max_{v_t, z_t} \ln(c_t) - \gamma(y_t + z_t) + \beta P(k_{t+1}, y_{t+1}, t+1) \\ \text{s.t.} & \quad (8.7), (8.9), \\ & \quad 0 \leq z_t \leq 1 - y_t, \\ & \quad 0 \leq v_t \leq 1. \end{aligned}$$

The first-order and envelope conditions reduce to a first-order condition for capital accumulation which corresponds to (8.1) and (8.4), as well as to

$$\frac{f_y(A_t, k_t, y_t(1 - v_t))}{c_t} = -\gamma \frac{\pi'(\theta_t)\theta_t + \pi(\theta_t)}{\pi'(\theta_t)\theta_t^2}, \quad (8.12)$$

$$\gamma = -\beta\pi'(\theta_t)\theta_t^2\Phi_{t+1}, \quad (8.13)$$

where we define (dropping again arguments)

$$\Phi_{t+1} \equiv \frac{f_y(t+1)}{c_{t+1}} - \gamma + \gamma \frac{1-s}{-\pi'(\theta_{t+1})\theta_{t+1}^2}.$$

Note the $\pi'(\theta_t)$ terms in the planner's optimality conditions; they reflect that the planner internalizes the search externalities. Note also that $m_z(z_t, v_t) = -\pi'(\theta_t)\theta_t^2$ and $m_v(z_t, v_t) = \pi'(\theta_t)\theta_t + \pi(\theta_t)$.

Condition (8.12) states that the planner equalizes the relative costs and benefits of recruiting and job search: Recruiting generates $m_v(z_t, v_t)$ matches and has opportunity cost $f_y(A_t, k_t, y_t(1 - v_t))$ while job search generates $m_z(z_t, v_t)$ matches and has opportunity cost γc_t in consumption units. Condition (8.13) characterizes the efficient intensity of job search. It equalizes the utility cost of job search and the discounted, probability weighted benefit from a marginal match. This benefit reflects utility from consumption, net of the utility loss from working, plus the cost saving from reduced job search.

Let $\eta(\theta_t)$ denote the elasticity of the matching function with respect to job search,

$$\eta(\theta_t) \equiv \frac{m_z(z_t, v_t)}{m(z_t, v_t)} z_t = \frac{m_z(z_t, v_t)}{\pi(\theta_t)\theta_t}.$$

If the bargaining weight of the job seeker, ϕ , coincides with this elasticity that is, when the *Hosios* condition is satisfied, then the decentralized equilibrium allocation and the social planner allocation coincide and in particular, conditions (8.10)–(8.11) and (8.12)–(8.13) are identical.

This can be shown as follows: Using (8.11), the right-hand side of (8.10) can be expressed as $\gamma(1 - \phi)/(\phi\theta_t)$ and thus (using the Hosios condition and the CRTS property of the matching function), $\gamma m_v(z_t, v_t)/m_z(z_t, v_t)$. Moreover, using the modified condition (8.10), the last term of Ω_{t+1} can be expressed as $(1 - s)\gamma/(\phi\pi(\theta_{t+1})\theta_{t+1})$. The Hosios condition and the above definition of the elasticity then implies $\Omega_{t+1} = \Phi_{t+1}$.

Intuitively, the Hosios condition guarantees that the private gains from job seeking or posting vacancies (which households or firms internalize) equal the social contributions of these activities (which they do not internalize). When the elasticity $\eta(\theta_t)$ is high then job search strongly increases the odds that a vacancy is filled but it only has a minor negative effect on the probability that a job seeker is matched. In contrast, recruiting strongly reduces the odds that a vacancy is filled in this case while it only has a minor positive effect on the job finding probability of a job seeker. When $\eta(\theta_t)$ is high it is therefore efficient to give strong incentives for job seeking that is, pay high wages.

8.2.7 Analysis

Returning to the decentralized equilibrium consider a variant of the model without capital that is, let $f(A_t, k_t, y_t(1 - \nu_t)) = A_t y_t(1 - \nu_t)$. The first-order conditions for ν_t and z_t , the wage condition, the resource constraint, and the budget constraint then reduce to

$$\begin{aligned}\frac{A_t}{c_t} &= \beta\pi(\theta_t) \left(\frac{A_{t+1}}{c_{t+1}} \left(1 + \frac{1-s}{\pi(\theta_{t+1})} \right) - \frac{w_{t+1}}{c_{t+1}} \right), \\ \gamma &= \beta\theta_t\pi(\theta_t) \left(\frac{w_{t+1}}{c_{t+1}} - \gamma + \gamma \frac{1-s}{\theta_{t+1}\pi(\theta_{t+1})} \right), \\ \frac{w_t}{c_t} &= \gamma \left(1 - \frac{1-s}{\theta_t\pi(\theta_t)} \right) + \phi \left\{ \frac{A_t}{c_t} \left(1 + \frac{1-s}{\pi(\theta_t)} \right) - \gamma \left(1 - \frac{1-s}{\theta_t\pi(\theta_t)} \right) \right\}, \\ \frac{c_t}{A_t} &= y_t(1 - \nu_t),\end{aligned}$$

respectively.

Note that productivity, consumption, and the wage only appear as ratios in these conditions. Productivity fluctuations thus are reflected in consumption and wages but not in employment, recruiting, job search, and labor market tightness. For example, let $(A, w, c, y, z, \nu, \theta)$ denote steady-state values and consider a fluctuating productivity sequence, $\{A_t\}_{t \geq 0}$; let $\xi_t \equiv A_t/A$. The equilibrium conditions imply that conditional on $\{A_t\}_{t \geq 0}$, the sequences

$$\{w_t, c_t, y_t, z_t, \nu_t, \theta_t\}_{t \geq 0} = \{\xi_t w, \xi_t c, y, z, \nu, \theta\}_{t \geq 0}$$

constitute an equilibrium.

Intuitively, holding wages and consumption (and thus, interest rates) constant, a contemporaneous productivity increase renders recruiting more expensive relative to production, thereby inducing firms to recruit less and produce more (see the first condition). But higher production raises consumption, and less recruiting lowers future production and consumption. In equilibrium, this requires a fall in the interest rate (from the Euler equation), raising the incentive to invest in vacancies. With logarithmic preferences, the direct, negative effect on the incentive to recruit and the indirect, positive effect cancel.

Capital accumulation changes this result since capital cannot instantaneously adjust in line with productivity. But in steady state the above argument continues to apply. In a steady state with higher productivity, the marginal product of capital is unchanged but the marginal product of labor, the capital stock, the wage, and consumption are higher as well. In contrast, steady-state employment, job search, and recruiting remain unchanged.

Wage stickiness renders labor market outcomes more responsive to productivity. When firms anticipate elevated productivity but unchanged wages then recruiting becomes more profitable (see the optimality condition for ν_t above). Employment therefore rises more strongly than in an environment with Nash bargaining where wages

increase with productivity. Wage stickiness is individually rational in the sense that it does not give rise to inefficiencies from the joint perspective of a household and a firm as long as the wage remains within the bargaining set delimited by the pair's reservation wages.

The *Beveridge curve* depicts combinations of steady-state unemployment and vacancies. To derive the curve we simplify the model further, abstracting from leisure. Household members thus are either employed or unemployed (searching for a job), $z_t = 1 - y_t$. In steady state, employment is constant and the law of motion (8.9) implies $m(z, v) = s(1 - z)$ or

$$z = \frac{s}{s + \theta\pi(\theta)},$$

where $\theta \equiv v/z$. Since $\theta\pi(\theta)$ is increasing and concave the Beveridge curve is decreasing and convex in (θ, z) space. The same holds true in (v, z) space.

The Beveridge curve is decreasing because both job search and vacancies increase the number of matches. In steady state, net inflows into employment equal zero. Additional vacancies thus must be accompanied by fewer job seekers, both to reduce the number of matches and to increase the number of outflows from employment, $s(1 - z)$. If the matching technology improves then the Beveridge curve shifts inward—*structural unemployment* and vacancies fall; if the separation rate, s , rises then the Beveridge curve shifts out.

Long-run equilibrium in the labor market is determined by the intersection of the Beveridge curve and a schedule representing the other equilibrium conditions. Equilibrium unemployment is strictly positive; if it equalled zero firms would not recruit.

8.3 Credit Frictions

8.4 Bibliographic Notes

Tobin (1969) discusses the relative price of installed capital and Hayashi (1982) analyzes the relation between marginal and average q . Lucas and Prescott (1971) and Dixit (1989, 1–2) study risky investment decisions and the option value of waiting.

Diamond (1982), Mortensen (1982), and Pissarides (1985), among others, develop the search and matching model of the labor market. Pissarides (1990; 2000) describes the baseline model and reviews the literature. The presentation in the text follows Shimer (2010) who builds on Merz (1995). Hosios (1990) analyzes constrained efficiency and Hall (2005) proposes the model of individually-rational sticky wages.

Chapter 9

Nominal Assets and Money

Chapter 10

Price Setting, Sticky Prices, and Demand Determined Output

Chapter 11

The Government

Fiscal and monetary policies affect budget constraints, change incentives, and absorb resources. To analyze these effects, we introduce a government sector. Our equilibrium concept remains unchanged: Households and firms optimize, and markets clear. But the presence of the government sector requires that we refine this concept along three dimensions. First, we specify that agents in the private sector take the government's policy as given, in the same way as households and firms in a competitive equilibrium take prices as given. Second, we require that the government also satisfies its budget constraints. And third, we account for the government's resource use when specifying resource constraints or market clearing conditions. When a specific policy implements an equilibrium so defined then we say that the policy is *feasible*.

We analyze the macroeconomic effects of taxation, government consumption, debt and social security; identify conditions under which policy changes are neutral; and study how fiscal and monetary policy jointly affects inflation.

11.1 Taxation and Government Consumption

Consider the representative agent economy discussed in section 3.1, augmented by a government that levies taxes on labor income, at rate τ_t^w , and on the return on saving, at rate τ_t^k , to finance government spending.¹ The budget constraint of a household reads

$$a_{t+1} = a_t R_t (1 - \tau_t^k) + w_t (1 - \tau_t^w) - c_t$$

and household optimization thus gives rise to the Euler equation

$$u'(c_t) = \beta R_{t+1} (1 - \tau_{t+1}^k) u'(c_{t+1}).$$

The tax on capital income reduces the net return on household saving (or borrowing). A higher τ_{t+1}^k therefore induces the same type of income and substitution effects as

¹To simplify notation, we assume in this chapter that the latter tax is levied on the gross return that is, the principal is taxed as well. If, more realistically, the tax only were levied on income, at rate θ_t say, then the budget constraint would read $a_{t+1} = a_t (1 + (r_t - \delta)(1 - \theta_t)) + w_t (1 - \tau_t^w) - c_t$. Clearly, $\theta_t = \tau_t^k + \tau_t^k / (r_t - \delta)$.

a lower R_{t+1} (see subsection 2.1.1), and it discourages saving. In contrast, the labor income tax does not induce a substitution effect since labor is supplied inelastically; it only reduces household wealth.

Taxes finance government consumption, g_t , which may or may not be valued by households. We assume that it is not valued, or that preferences are separable between private and government consumption such that the household's first-order conditions are unaffected by g_t . For now, we abstract from government deficits or surpluses. This implies the government budget constraint

$$g_t = a_t R_t \tau_t^k + w_t \tau_t^w.$$

Substituting into the household's constraint yields

$$a_{t+1} = a_t R_t + w_t - c_t - g_t,$$

indicating that taxation reduces disposable income.

Combining the budget constraints of households, firms, and the government and imposing the market clearing and equilibrium conditions discussed in section 3.1 yields the core equilibrium conditions

$$\begin{aligned} k_{t+1} &= k_t(1 - \delta) + f(k_t, 1) - c_t - g_t, \\ u'(c_t) &= \beta(1 + f_K(k_{t+1}, 1) - \delta)(1 - \tau_{t+1}^k)u'(c_{t+1}). \end{aligned}$$

Compared to the core conditions in the model without government, (3.8) and (3.9), the resource constraint (or GDP identity) now accounts for government consumption, and the tax rate on capital income enters the Euler equation.

Consider first a steady state which is characterized by the conditions

$$\begin{aligned} c &= f(k, 1) - \delta k - g, \\ 1 &= \beta(1 + f_K(k, 1) - \delta)(1 - \tau^k). \end{aligned}$$

The resource constraint states that replacement investment, δk , and total consumption, $c + g$, equal output that is, conditional on the capital stock, private consumption falls one-to-one with government consumption. The Euler equation states that the after-tax return on saving equals β^{-1} ; a tax on capital income—but not on labor income—therefore reduces the capital stock. Accordingly, steady-state private consumption is maximal (conditional on g) if the government only levies labor income taxes.

Off steady state, capital income taxation generates inferior outcomes too. To see this, note that the equilibrium conditions with labor but no capital income taxation are identical to the conditions in a model where households pay a lump-sum tax equal to g_t . The latter conditions, in turn, correspond to the optimality conditions in a Robinson Crusoe economy where only the resource constraint binds and the “government” consumes g_t . This implies that the equilibrium allocation with labor income taxes is Pareto optimal, in contrast to the allocation with capital income taxes.

While labor and capital income taxes generate the same revenue, the two instruments have different incentive effects since only capital income taxes can be avoided.

From the perspective of an individual household that takes tax rates as given and only internalizes the cost of paying taxes, a capital income tax discourages saving. The benefit of paying taxes—a lower equilibrium tax rate for everybody, given that total tax revenue must equal g_t —is not internalized. This drives a *wedge*, $1 - \tau_{t+1}^k$, between the social marginal rate of transformation, R_{t+1} , and the private marginal rate of transformation, $R_{t+1}(1 - \tau_{t+1}^k)$, which the household equalizes with the private marginal rate of substitution.

These findings generalize. Not only does government consumption reduce household wealth, but its financing by means of taxes that induce substitution effects gives rise to welfare reducing *tax distortions*. Abstracting from distributive implications (which are absent in this environment with homogeneous households), taxes that induce substitution effects (here, capital income taxes) therefore generate Pareto inferior outcomes than taxes that do not induce such effects (here, labor income taxes).

In endogenous growth models of the type considered in subsection 6.2.2, a tax induced reduction in the after-tax interest rate lowers the economy's equilibrium *growth rate*, with potentially large welfare consequences.

11.2 Government Debt and Social Security

Next, we introduce government debt as a source of funding. We restrict the analysis to non-distorting taxation of labor income at rate τ_t and allow for population growth at gross rate ν . The resource constraint is given by

$$\nu k_{t+1} = f(k_t, 1) + k_t(1 - \delta) - c_t - g_t,$$

where capital as well as private and government consumption are expressed in per-worker terms.

Government debt allows to intertemporally decouple tax collections and government spending. The dynamic budget constraint of the government reads

$$\nu b_{t+1} = b_t R_t + g_t - \tau_t w_t,$$

where b_t denotes the stock of government debt per worker. The constraint states that a primary deficit, $g_t - \tau_t w_t > 0$, must be financed by debt issuance in excess of debt service (repayment of principal plus interest). Equivalently, a deficit, $b_t(R_t - 1) + g_t - \tau_t w_t > 0$, increases the government's indebtedness. Note that we do not distinguish between the interest rates on government debt and capital. Market clearing requires that households are indifferent between the two assets and thus, absent risk, that the interest rates coincide.

Let $q_0 \equiv 1$ and $q_t \equiv (R_1 \cdots R_t)^{-1}$, $t > 0$. If the no-Ponzi-game condition $\lim_{t \rightarrow \infty} q_t \nu^{t+1} b_{t+1} \leq 0$ is imposed with equality, then the dynamic budget constraints imply the intertemporal government budget constraint

$$0 = b_0 R_0 + \sum_{t=0}^{\infty} q_t \nu^t (g_t - \tau_t w_t).$$

11.2.1 Government Debt with a Representative Agent

Consider first the representative agent model. The representative family maximizes $\sum_{t=0}^{\infty} \beta^t v^t u(c_t)$ subject to the dynamic budget constraint and a no-Ponzi-game condition. The Euler equation, dynamic and intertemporal budget constraints are given by

$$\begin{aligned} u'(c_t) &= \beta R_{t+1} u'(c_{t+1}), \\ v a_{t+1} &= a_t R_t + w_t(1 - \tau_t) - c_t, \\ 0 &= a_0 R_0 + \sum_{t=0}^{\infty} q_t v^t (w_t(1 - \tau_t) - c_t). \end{aligned}$$

Suppose the government balances its budget in each period such that $b_t = 0$ and $\tau_t w_t = g_t$, implying $k_t = a_t$. Consider the following change of financing policy, keeping the path of government consumption unchanged: The government alters the timing of tax collections while holding the present value of taxes (at the initial interest rates) constant. For example, the government reduces tax collections at date t by Δ per capita and increases taxes at date $t + 1$ by $\Delta R_{t+1}/v$ per capita. The government also issues debt Δ per capita at date t and fully services it in the subsequent period out of the additional taxes raised.

Capital accumulation, consumption, interest rates and wages are not affected by this policy change. To see this, we conjecture that wages and interest rates indeed remain unchanged and we verify that the initial capital and consumption sequences then continue to satisfy all equilibrium conditions. Under the conjecture, the representative family's budget set is unaffected by the policy change because

$$-\Delta + \frac{\Delta R_{t+1}}{v} \frac{v}{R_{t+1}} = 0.$$

The original consumption sequence therefore remains optimal. From the household's dynamic budget constraint, this implies that the family increases saving when the government runs a deficit, by exactly the same amount. Since $a_{t+1} = b_{t+1} + k_{t+1}$, capital accumulation and thus, wages and interest rates therefore remain unchanged. Since the government's budget constraints are satisfied as well the conjecture is verified. We conclude that the equilibrium allocation (except for government debt) remains unchanged.

This result is an instance of the *Ricardian equivalence* proposition. The proposition states that for a given government consumption sequence (and thus, present discounted value of taxes) the timing of tax collections does not affect the equilibrium allocation. Note that the proposition makes a statement about changes in government financing, not government consumption. The proposition holds under three key assumptions. First, households and the government can save or borrow at the same interest rates. Second, the policy change does not shift the tax burden from one group to another. (With a representative family this is satisfied by assumption.) And third, taxes are not distorting. These assumptions guarantee that a change of government financing policy does not alter budget sets in the private sector.

11.2.2 Government Debt with Overlapping Generations

Consider next the two-period lived OLG model. In general equilibrium, we have

$$\begin{aligned} u'(c_{1,t}) &= \beta R_{t+1} u'(c_{2,t+1}), \\ v(b_{t+1} + k_{t+1}) &= w_t(1 - \tau_t) - c_{1,t}, \\ c_{2,t+1} &= v(b_{t+1} + k_{t+1})R_{t+1}, \end{aligned}$$

where $c_{1,t}$ and $c_{2,t}$ denote consumption of a young and old household at date t , respectively. Note that we have imposed the market clearing condition $a_{t+1} = b_{t+1} + k_{t+1}$.

In this economy, Ricardian equivalence does not hold. Reducing taxes at date t and increasing them at date $t + 1$ shifts the tax burden from workers in cohort t to those in the subsequent cohort. The debt issued to finance the government deficit and acquired by workers at date t constitutes net wealth for this group in the sense that they do not have to contribute future resources to service it. Because of the tax cut's positive wealth effect on cohort t , the workers increase their saving by less than the amount of the tax cut, raising consumption.

Capital accumulation therefore slows down: Government debt *crowds out* capital. As a consequence, interest rates rise and wages fall in the subsequent period. Cohort t does not only benefit from a lighter tax burden but also from a higher return on saving while cohort $t + 1$ bears a heavier tax burden and receives lower wages.

Along a balanced growth path with constant per-capita values, the government budget constraint reads

$$\nu b = bR + g - \tau w.$$

Absent population growth, taxes equal government consumption and interest payments on debt; with population growth, taxes fall short of these spending items because new debt is issued in each period. If the economy is dynamically inefficient, $R < \nu$, then the revenue raised from new debt issuance exceeds interest payments and the government may even purchase goods without ever collecting taxes, simply by holding the debt-to-worker ratio constant ($g > 0, \tau = 0, b > 0$; note that this violates the no-Ponzi-game condition). Debt is welfare increasing under these circumstances because it reduces capital over accumulation. Moreover, debt is a *bubble*: While it never generates dividends (tax revenues) it is rolled over forever at a positive price.

11.2.3 Pay-As-You-Go Social Security with Overlapping Generations

Maintaining the overlapping generations structure, consider finally a *pay-as-you-go* social security system. In contrast to a *fully funded* system where households contribute resources into accounts and consume the accumulated return after retirement (and where in general equilibrium, the contributions fund capital accumulation) the contributions in a pay-as-you-go system finance benefits to the contemporaneous retirees. That is, the pay-as-you-go system is a transfer system. Each old household at date t receives a transfer, $T_t \nu$, that is fully financed by labor income taxes levied at rate τ_t^s , such that $T_t = w_t \tau_t^s$. There is no debt and for simplicity, we let $g_t = 0$. In general

equilibrium,

$$\begin{aligned} u'(c_{1,t}) &= \beta R_{t+1} u'(c_{2,t+1}), \\ \nu k_{t+1} &= w_t(1 - \tau_t^s) - c_{1,t}, \\ c_{2,t+1} &= \nu k_{t+1} R_{t+1} + T_{t+1} \nu. \end{aligned}$$

For any feasible social security policy $\{\tau_t^s, T_t\}_{t \geq 0}$, there exists an equivalent tax-and-debt policy $\{\tau_t, b_{t+1}\}_{t \geq 0}$ that implements the same equilibrium allocation. To see this, note first that under the social security policy, the intertemporal budget constraint of a household in cohort t is given by

$$c_{1,t} + \frac{c_{2,t+1}}{R_{t+1}} = w_t(1 - \tau_t^s) + \frac{T_{t+1}\nu}{R_{t+1}},$$

while under the tax-and-debt policy, the constraint reads

$$c_{1,t} + \frac{c_{2,t+1}}{R_{t+1}} = w_t(1 - \tau_t).$$

Given the equilibrium prices implemented by the social security policy, the budget sets characterized by the two constraints are identical if the present value of taxes net of transfers under the social security policy, $w_t \tau_t^s - T_{t+1} \nu / R_{t+1}$, equals taxes under the tax-and-debt policy, $w_t \tau_t$. Since $T_{t+1} = w_{t+1} \tau_{t+1}^s \nu$, this implies the first equivalence condition,

$$\tau_t = \tau_t^s - \frac{w_{t+1} \tau_{t+1}^s \nu}{w_t R_{t+1}},$$

which maps the sequence of social security tax rates into a sequence of tax rates.

A second equivalence condition follows from the requirement that the dynamic budget constraints of the government or households be satisfied. The two policies pay the same amount of funds to the old if

$$b_t R_t = w_t \tau_t^s,$$

which maps the sequence of social security tax rates into a debt sequence. Since neither policy affects the resource constraint or the factor price conditions we conclude that the two equivalence conditions map any feasible social security policy into a tax-and-debt policy that implements the same equilibrium allocation and prices. Absent restrictions on the available tax and transfer instruments, similar mappings can be derived in environments where social security taxes are distorting, households long-lived and heterogenous within a cohort, or outcomes stochastic.

Intuitively, under the social security policy, households save little because they receive transfers in old age. Under the equivalent tax-and-debt policy, they save more because they pay lower taxes when young but do not receive transfers when old. The difference in saving exactly corresponds to the debt the government issues under the tax-and-debt policy. In light of this equivalence, one refers to the present discounted

value of the already committed future social security benefits as the *implicit debt* of the pay-as-you-go financed social security system.

Since the implicit debt associated with a social security policy and the explicit debt associated with an equivalent tax-and-debt policy entail the same financial commitments, focusing on the latter and disregarding the former can be misleading. For example, explicit debt does not comprehensively measure the fiscal burden a policy imposes on future generations as these generations also have to contribute resources to service the implicit debt (unless the government defaults on the latter).

Generational accounts do provide a comprehensive measure. The *generational account* of a group is the present discounted value of that group's remaining lifetime net taxes. From the government's intertemporal budget constraint, the sum of all generational accounts equals the present discounted value of current and future government consumption plus outstanding government debt. Generational accounts thus account for commitments both due to explicit and implicit liabilities.

Suppose that at date $t = 0$ a pay-as-you-go social security system $\{\tau_t^s, T_t\}_{t \geq 0}$ is introduced. The effect on the budget set of an old household at date $t = 0$ and on the budget set of a member of cohort $t \geq 0$, respectively, are given by

$$w_0 \tau_0^s \nu \quad \text{and} \quad -w_t \tau_t^s + \frac{w_{t+1} \tau_{t+1}^s \nu}{R_{t+1}}.$$

Using the relations derived earlier, we can represent this in terms of an equivalent policy that finances a transfer to the old at date $t = 0$ out of taxes and debt which subsequent cohorts service over time, see table 11.1. The first generation receiving social security benefits clearly is made better off. Whether subsequent generations benefit or lose depends on whether the equilibrium is dynamically efficient or not. Along a dynamically inefficient balanced growth path ($R < \nu$) subsequent generations also benefit because

$$-w \tau^s + \frac{w \tau^s \nu}{R} > 0.$$

Along a dynamically efficient balanced growth path, in contrast, they are harmed.

Beyond the deterministic OLG model, a state-contingent social security policy (or equivalent tax-and-debt policy with state-contingent returns on government debt) may contribute to inter generational risk sharing. This can be valuable because, absent such policies, overlapping generations cannot implement all ex-ante beneficial insurance arrangements. Moreover, a state-contingent social security policy that provides annuities may also contribute to intra generational risk sharing by insuring longevity risk.

11.3 Equivalence of Policies

The Ricardian equivalence proposition discussed in subsection 11.2.1 describes *equivalence classes* of fiscal policies whose members implement the same equilibrium allocation that is, the same sequences for consumption, capital, wages, and interest rates but not necessarily for financial assets like government debt. Our discussion of equivalent

	Pay-as-you-go	Explicit debt
<i>Effect on household budget at date t</i>		
lifetime net taxes:	τ_t	= τ_t
+ taxes on young households	τ_t^s	> τ_t
– discounted old age benefits	$\frac{T_{t+1}v_{t+1}}{R_{t+1}}$	> 0
<i>Effect on government budget at date t</i>		
cash flow, $t = 0$:	0	= 0
+ total cash inflow, $t = 0$	$N_0\tau_0^s$	= $N_0\tau_0^s$
+ taxes on young households, $t = 0$	$N_0\tau_0^s$	> $N_0\tau_0$
+ debt issued, $t = 0$	0	< $N_0b_1v_1$
– total cash outflow, $t = 0$	N_0T_0	= $N_0\theta_0$
– transfer to old households, $t = 0$	N_0T_0	= $N_0\theta_0$
cash flow, $t > 0$:	0	= 0
+ total cash inflow, $t > 0$	$N_t\tau_t^s$	= $N_t\tau_t^s$
+ taxes on young households, $t > 0$	$N_t\tau_t^s$	> $N_t\tau_t$
+ debt issued, $t > 0$	0	< $N_tb_{t+1}v_{t+1}$
– total cash outflow, $t > 0$	N_tT_t	= $N_tb_tR_t$
– transfer to old households, $t > 0$	N_tT_t	> 0
– debt service, $t > 0$	0	< $N_tb_tR_t$

Note: N_t denotes the size of cohort t and $v_{t+1} \equiv N_{t+1}/N_t$ its possibly time-varying growth rate. Wages are normalized to one. Equivalence then requires $v_{t+1}b_{t+1} = \tau_t^s - \tau_t$ and $T_{t+1} = b_{t+1}R_{t+1}$. In the economy with government debt, the transfer to the initial old, θ_0 , corresponds with the transfer paid under the pay-as-you-go system.

Table 11.1: Equivalence of pay-as-you-go social security and explicit government debt.

pay-as-you-go social security and tax-and-debt policies in subsection 11.2.3 identified another type of equivalence classes. We now unify these discussions and present additional applications.

11.3.1 General Equivalence Result

Let μ denote the state at the initial date and let φ denote a policy. Equivalence classes relate pairs of policies and states. A pair (μ, φ) and another pair $(\hat{\mu}, \hat{\varphi})$ belong to the same equivalence class if and only if both pairs implement the same equilibrium allocation.²

A direct approach to establishing that (μ, φ) and $(\hat{\mu}, \hat{\varphi})$ belong to the same equivalence class relies on characterizing the equilibrium allocations implemented by each

²For simplicity, we disregard issues related to multiplicity of equilibria.

pair (if they exist) and showing that they are identical. An indirect approach relies on establishing that the choice sets of households and firms are not affected by the change of policy. Suppose a pair (μ, φ) implements an equilibrium and suppose that another pair, $(\hat{\mu}, \hat{\varphi})$, satisfies the following conditions:

- i. μ and $\hat{\mu}$ encode identical production possibilities, and restrictions on inputs and/or outputs of firms are identical across policies;
- ii. households' choice sets are identical if evaluated at the equilibrium prices;
- iii. at the equilibrium allocation and prices, $(\hat{\mu}, \hat{\varphi})$ satisfies the government's dynamic budget constraints.

The two pairs then belong to the same equivalence class.

This can be seen as follows: Conjecture that equilibrium prices under (μ, φ) and (μ', φ') are the same. With household choice sets unchanged, household demand functions are unaltered since preferences do not depend on policy. With constraints on production unaffected, firm net supply functions are unaltered. The original household and firm choices (except possibly for financial assets) thus remain optimal, clear markets, and continue to satisfy the resource constraints. Private sector choices and the government's new policy also satisfy the relevant budget constraints. Given that the equilibrium allocation under (μ, φ) and (μ', φ') is the same, the conjecture is verified.

11.3.2 Applications

Note that the reasoning supporting the general equivalence result parallels the arguments establishing Ricardian equivalence as well as equivalence of pay-as-you-go social security and tax-and-debt policies. There, the choice set of a household is the set of affordable consumption allocations over the household's life, and condition i. is trivially satisfied because the initial capital stock which corresponds to μ is held constant. But the result applies much more broadly as the following examples show.

Heterogeneity Suppose that households are heterogeneous within cohorts. An equivalence class (conditional on some initial state) then consists of policies that satisfy the government budget constraints and impose on each household a given household specific present discounted value of taxes.

Tax Distortions Suppose that labor income taxes are distorting. The choice set of a household then is given by the set of affordable consumption and leisure allocations. An equivalence class (conditional on the initial state) consists of policies that satisfy the government budget constraints and impose on each household a given household specific lifetime *tax function* which specifies the present discounted value of taxes as a function of the household's choices. For example, one tax policy in such an equivalence class might tax labor income at date t at rate τ_t^w , while another policy in the same class might tax labor income at date t at rate $\tau_t^w R_{t+1}$ but collect the tax only in the

subsequent period. Since both policies have the same effect on the household's choice set an equilibrium allocation implemented by the former policy also constitutes an equilibrium allocation under the latter. As with "standard" Ricardian equivalence, however, the two policies are associated with different levels of government debt.

Multiple Tax Instruments Suppose that the government taxes consumption expenditures at rate τ_t^c , capital income at rate τ_t^k , and labor income at rate τ_t^w . The household's dynamic budget constraint reads

$$a_{t+1} = a_t R_t (1 - \tau_t^k) + w_t (1 - x_t) (1 - \tau_t^w) - c_t (1 + \tau_t^c),$$

where x_t denotes leisure. Integrating the dynamic budget constraints and imposing a no-Ponzi-game condition yields the intertemporal budget constraint

$$a_0 R_0 (1 - \tau_0^k) + \sum_{t=0}^{\infty} q_t \kappa_t (w_t (1 - x_t) (1 - \tau_t^w) - c_t (1 + \tau_t^c)) = 0,$$

where we define $\kappa_0 \equiv 1$ and $\kappa_t \equiv [(1 - \tau_1^k) \cdots (1 - \tau_t^k)]^{-1}$, $t > 0$. Letting $\xi_t \equiv (1 + \tau_t^c) / (1 + \tau_0^c)$, we can rewrite this as

$$\frac{a_0 R_0 (1 - \tau_0^k)}{1 + \tau_0^c} + \sum_{t=0}^{\infty} q_t \kappa_t \xi_t \left(w_t (1 - x_t) \frac{1 - \tau_t^w}{1 + \tau_t^c} - c_t \right) = 0.$$

Note that from the household's perspective, the price at date t of leisure relative to consumption equals $w_t (1 - \tau_t^w) / (1 + \tau_t^c)$ and the price of consumption at date $t + 1$ relative to consumption at date t equals $q_{t+1} \kappa_{t+1} \xi_{t+1} / (q_t \kappa_t \xi_t) = (1 + \tau_{t+1}^c) / (R_{t+1} (1 - \tau_{t+1}^k) (1 + \tau_t^c))$. That is, the *tax wedges*

$$\frac{1 - \tau_t^w}{1 + \tau_t^c} \quad \text{and} \quad \frac{1 + \tau_{t+1}^c}{(1 - \tau_{t+1}^k) (1 + \tau_t^c)}$$

distort the consumption-leisure and consumption-saving choices, respectively.

If $a_0 R_0 = 0$ then only the two tax wedges, not the three tax rates individually, affect the household's budget set. Feasible tax sequences generating the same wedge sequences therefore form an equivalence class in this case. For example, a feasible tax policy employing all three tax instruments is equivalent to another policy that only relies on a specific combination of capital and labor income taxes.

If $a_0 R_0 \neq 0$ then the budget set also depends on $(1 - \tau_0^k) / (1 + \tau_0^c)$, in addition to the two wedge sequences. Note that a capital income tax levied on pre-determined financial wealth can be replicated by a consumption tax in the initial period and, keeping the two wedge sequences unchanged, higher consumption tax rates in all subsequent periods.

11.4 Nominal Government Debt and Money

Governments issue nominal debt and central bank money. This implies that the government budget constraint links fiscal and monetary policy instruments as well as the price level. The fact that money carries low or no interest also gives rise to a novel revenue source, seignorage.

11.4.1 Consolidated Government Budget Constraint

The government at date t , history ϵ^t collects taxes net of government consumption, $\tau_t(\epsilon^t) - g_t(\epsilon^t)$ (the primary surplus, both expressed in real terms); redeems maturing real and nominal debt, $b_t(\epsilon^{t-1})$ and $B_t(\epsilon^{t-1})$ respectively; and issues new real and nominal debt as well as additional money balances, $b_{t+1}(\epsilon^t)$, $B_{t+1}(\epsilon^t)$ and $M_{t+1}(\epsilon^t) - M_t(\epsilon^{t-1})$. Note that liabilities issued at date t and maturing at date $t + 1$ are indexed by the history ϵ^t . A unit of real or indexed debt issued at date t pays the potentially state-contingent gross return $R_{t+1}(\epsilon^{t+1})$, expressed in real terms. A unit of nominal debt pays the potentially state-contingent gross return $I_{t+1}(\epsilon^{t+1})$, expressed in nominal terms, which translates into the inflation adjusted rate of return $I_{t+1}(\epsilon^{t+1})\Pi_{t+1}^{-1}(\epsilon^{t+1})$, where $\Pi_{t+1}(\epsilon^{t+1})$ denotes the gross inflation rate. Throughout, debt positions should be interpreted as net debt positions of the government.

Let $\{m_{t+1}(\epsilon^{t+1})\}_{t \geq 0}$ denote the asset pricing kernel. The standard asset pricing condition

$$\mathbb{E}_t \left[m_{t+1}(\epsilon^{t+1}) \frac{\text{payoff}_{t+1}(\epsilon^{t+1})}{\text{price}_t(\epsilon^t)} \right] = 1$$

implies that the equilibrium price of real and nominal debt equals unity and $P_t^{-1}(\epsilon^t)$ (the inverse of the price level), respectively. For if the two types of debt pay the required real rates of return then

$$\begin{aligned} \mathbb{E}_t \left[m_{t+1}(\epsilon^{t+1}) \frac{R_{t+1}(\epsilon^{t+1})}{1} \right] &= 1, \\ \mathbb{E}_t \left[m_{t+1}(\epsilon^{t+1}) \frac{I_{t+1}(\epsilon^{t+1})/P_{t+1}(\epsilon^{t+1})}{1/P_t(\epsilon^t)} \right] &= 1. \end{aligned}$$

Moreover, when issuing $M_{t+1}(\epsilon^t) - M_t(\epsilon^{t-1})$ units of money at date t , the government receives $1/P_t(\epsilon^t)$ units of the good in exchange. Combined, these results imply that the government's consolidated dynamic budget constraint at date t , history ϵ^t reads

$$\begin{aligned} b_t(\epsilon^{t-1})R_t(\epsilon^t) + \frac{B_t(\epsilon^{t-1})I_t(\epsilon^t)}{P_t(\epsilon^t)} &= \\ \tau_t(\epsilon^t) - g_t(\epsilon^t) + b_{t+1}(\epsilon^t) + \frac{B_{t+1}(\epsilon^t)}{P_t(\epsilon^t)} + \frac{M_{t+1}(\epsilon^t) - M_t(\epsilon^{t-1})}{P_t(\epsilon^t)}. \end{aligned} \quad (11.1)$$

Note that the real value of government debt including interest on the left-hand side of the equation may be state-contingent for two reasons: Because interest rates vary with the state of nature or, with nominal debt, because the price level is stochastic.

Solving the dynamic budget constraint forward and imposing a no-Ponzi-game condition yields

$$b_t(\epsilon^{t-1})R_t(\epsilon^t) + \frac{B_t(\epsilon^{t-1})I_t(\epsilon^t)}{P_t(\epsilon^t)} = \quad (11.2)$$

$$\sum_{j=0}^{\infty} \mathbb{E}_t \left[(m_{t+1}(\epsilon^{t+1}) \cdots m_{t+j}(\epsilon^{t+j})) \times \left(\tau_{t+j}(\epsilon^{t+j}) - g_{t+j}(\epsilon^{t+j}) + \frac{M_{t+1+j}(\epsilon^{t+j}) - M_{t+j}(\epsilon^{t+j-1})}{P_{t+j}(\epsilon^{t+j})} \right) \right].$$

Equation (11.2) states that the market value of outstanding debt equals the present discounted value of current and future primary surpluses including seignorage revenues, where *seignorage* is defined as the resources the government collects in exchange for the money it issues.

If we define government liabilities more broadly including debt and outstanding money balances the dynamic budget constraint rewritten as

$$b_t(\epsilon^{t-1})R_t(\epsilon^t) + \frac{B_t(\epsilon^{t-1})I_t(\epsilon^t)}{P_t(\epsilon^t)} + \frac{M_t(\epsilon^{t-1})}{P_t(\epsilon^t)} = \tau_t(\epsilon^t) - g_t(\epsilon^t) + b_{t+1}(\epsilon^t) + \frac{B_{t+1}(\epsilon^t)}{P_t(\epsilon^t)} + \frac{M_{t+1}(\epsilon^t)}{P_t(\epsilon^t)}$$

can be solved forward to yield

$$b_t(\epsilon^{t-1})R_t(\epsilon^t) + \frac{B_t(\epsilon^{t-1})I_t(\epsilon^t)}{P_t(\epsilon^t)} + \frac{M_t(\epsilon^{t-1})}{P_t(\epsilon^t)} = \sum_{j=0}^{\infty} \mathbb{E}_t \left[(m_{t+1}(\epsilon^{t+1}) \cdots m_{t+j}(\epsilon^{t+j})) \times \left(\tau_{t+j}(\epsilon^{t+j}) - g_{t+j}(\epsilon^{t+j}) + \frac{m_{t+1+j}(\epsilon^{t+1+j}) M_{t+1+j}(\epsilon^{t+j}) i_{t+1+j}(\epsilon^{t+1+j})}{P_{t+1+j}(\epsilon^{t+1+j})} \right) \right].$$

The last term on the right-hand side of the previous equation represents an alternative measure of seignorage, namely the cost reduction for the government due to the fact that money does not pay interest, in contrast to debt. This cost reduction enters the budget constraint in parallel to a tax revenue. If the government paid *interest on reserves* no such seignorage term would be present.

11.4.2 Seignorage Needs as Driver of Inflation

Consider a deterministic economy where the government does not issue debt and fixes taxes and government consumption at some exogenous values, $g - \tau > 0$. For an

equilibrium to exist, seignorage revenue then must be sufficient to balance the budget. Absent debt, equation (11.1) simplifies to

$$g - \tau = \frac{M_{t+1} - M_t}{P_t} = \frac{M_{t+1} - M_t}{M_t} \frac{M_t}{P_{t-1}} \frac{P_{t-1}}{P_t}.$$

Assume that M_t grows at the constant gross rate γ_M and the private sector's demand for real balances depends negatively on expected inflation. Expected and actual inflation coincide and equal the money growth rate, reflecting a quantity theory relation. This implies the equilibrium relationship

$$g - \tau = \frac{\gamma_M - 1}{\gamma_M} \cdot \text{money demand}(\gamma_M).$$

The right-hand side of this equation is hump shaped in γ_M since it is the product of an increasing and a decreasing function of γ_M : Higher money growth increases the tax rate on money balances but households respond by reducing the tax base, their money holdings. Because of this hump shaped *seignorage Laffer curve*, there exist two money growth rates that generate a given amount of seignorage unless that amount is too high. Policies that implement accelerating money growth and thus, inflation may generate even higher seignorage revenues.

11.4.3 A Simple Cash-in-Advance Economy

As a laboratory for our subsequent analysis, we use a simple endowment economy with an infinitely lived representative household and a government. Households own a state-contingent endowment sequence, $\{w_t(\epsilon^t)\}_{t \geq 0}$, and state-contingent government consumption equals $\{g_t(\epsilon^t)\}_{t \geq 0}$. An individual household may not consume its own endowment, and consumption goods can only be sold to, and bought from, other households against cash. The government must also transact using cash. The economy's resource constraint at date t , history ϵ^t reads $w_t(\epsilon^t) = c_t(\epsilon^t) + g_t(\epsilon^t)$. The households' and government's cash-in-advance constraints (which bind because of positive interest rates) are given by $c_t(\epsilon^t) = M_{t+1}^h(\epsilon^t)/P_t(\epsilon^t)$ and $g_t(\epsilon^t) = M_{t+1}^g(\epsilon^t)/P_t(\epsilon^t)$, respectively, with $M_{t+1}(\epsilon^t) \equiv M_{t+1}^h(\epsilon^t) + M_{t+1}^g(\epsilon^t)$. Securities are traded and money holdings are chosen after the state of nature is realized, before cash transactions take place.

The household's dynamic budget constraint reads

$$\begin{aligned} \tau_t(\epsilon^t) + b_{t+1}(\epsilon^t) + \frac{B_{t+1}(\epsilon^t)}{P_t(\epsilon^t)} + \frac{M_{t+1}^h(\epsilon^t)}{P_t(\epsilon^t)} = \\ b_t(\epsilon^{t-1})R_t(\epsilon^t) + \frac{B_t(\epsilon^{t-1})I_t(\epsilon^t)}{P_t(\epsilon^t)} + \frac{M_t^h(\epsilon^{t-1})}{P_t(\epsilon^t)} + \frac{w_{t-1}(\epsilon^{t-1}) - c_{t-1}(\epsilon^{t-1})}{\Pi_t(\epsilon^t)}, \end{aligned}$$

where the last term on the right-hand side represents the real value of cash inflows from endowment sales, net of cash outflows for consumption purchases in the previous

period. Since the cash-in-advance constraint binds, this collapses to

$$\begin{aligned} \tau_t(\epsilon^t) + b_{t+1}(\epsilon^t) + \frac{B_{t+1}(\epsilon^t)}{P_t(\epsilon^t)} + c_t(\epsilon^t) = \\ b_t(\epsilon^{t-1})R_t(\epsilon^t) + \frac{B_t(\epsilon^{t-1})I_t(\epsilon^t)}{P_t(\epsilon^t)} + w_{t-1}(\epsilon^{t-1})\Pi_t^{-1}(\epsilon^t). \end{aligned}$$

Note that positive inflation constitutes a tax on the endowment because the revenue from endowment sales which accrues in cash must be carried into the next period. The inflation tax is non-distorting, however, since the household cannot avoid it.

The initial state in the economy is $\mu = (M_0^h, M_0^g, b_0R_0, B_0I_0)$ and policy is given by

$$\varphi = \{\tau_t(\epsilon^t), g_t(\epsilon^t), b_{t+1}(\epsilon^t), R_{t+1}(\epsilon^{t+1}), B_{t+1}(\epsilon^t), I_{t+1}(\epsilon^{t+1}), M_{t+1}(\epsilon^t)\}_{t \geq 0}.$$

The endogenous variables are $\{c_t(\epsilon^t), P_t(\epsilon^t), m_{t+1}(\epsilon^{t+1}), M_{t+1}^h(\epsilon^t), M_{t+1}^g(\epsilon^t)\}_{t \geq 0}$. Equilibrium requires

$$\begin{aligned} c_t(\epsilon^t) &= w_t(\epsilon^t) - g_t(\epsilon^t), \\ m_{t+1}(\epsilon^{t+1}) &= \beta \frac{u'(w_{t+1}(\epsilon^{t+1}) - g_{t+1}(\epsilon^{t+1}))}{u'(w_t(\epsilon^t) - g_t(\epsilon^t))}, \\ \frac{M_{t+1}^h(\epsilon^t)}{M_{t+1}^g(\epsilon^t)} &= \frac{w_t(\epsilon^t) - g_t(\epsilon^t)}{g_t(\epsilon^t)}, \\ 1 &= \mathbb{E}_t \left[m_{t+1}(\epsilon^{t+1}) R_{t+1}(\epsilon^{t+1}) \right], \\ 1 &= \mathbb{E}_t \left[m_{t+1}(\epsilon^{t+1}) I_{t+1}(\epsilon^{t+1}) \Pi_{t+1}^{-1}(\epsilon^{t+1}) \right], \\ P_t(\epsilon^t) &= \frac{M_{t+1}(\epsilon^t)}{w_t(\epsilon^t)}, \end{aligned}$$

as well as the government budget constraints, conditions (11.1) and (11.2). Walras' Law implies that the household budget constraints then are satisfied as well.

Since policies within the same equivalence class implement the same allocation they must contain the same government consumption sequence, and since all other elements of a policy do not affect the equilibrium allocation, all feasible policies belong to the same equivalence class. Feasible policies may only differ from each other with respect to taxes, debt instruments, money balances, and interest rates. Different combinations of these instruments may give rise to different equilibrium price level sequences.

11.4.4 Inflation Effects of Government Financing

We now use the model of subsection 11.4.3 to study the implications of feasible policy changes for equilibrium inflation.

Irrelevance of Debt Composition Note first a neutrality result: A change of the composition of government debt between real and nominal debt accompanied by no change of taxes or money supply does not alter inflation. Such a policy change maintains the level of total indebtedness in real terms at each date and history and thus, total debt issuance in real terms. For example, a feasible policy φ with positive real and nominal debt implements the same equilibrium inflation as a modified policy $\hat{\varphi}$ with zero nominal debt and

$$\hat{b}_t(\epsilon^{t-1})\hat{R}_t(\epsilon^t) = b_t(\epsilon^{t-1})R_t(\epsilon^t) + \frac{B_t(\epsilon^{t-1})I_t(\epsilon^t)}{P_t(\epsilon^t)}, \quad t \geq 1.$$

Throughout the subsection, we may therefore abstract from nominal debt without loss of generality.

Policy Mixes We consider policy changes affecting taxes, seignorage and debt subject to (11.1), (11.2), the asset pricing condition, and the cash-in-advance constraint. Such changes generally do alter the equilibrium price level sequence. The policies before and after the change, φ and $\hat{\varphi}$ respectively, and the associated price level sequences, $\{P_t\}_{t \geq 0}$ and $\{\hat{P}_t\}_{t \geq 0}$, are related as follows:

$$\begin{aligned} \hat{\tau}_t(\epsilon^t) + \frac{\hat{M}_{t+1}(\epsilon^t) - \hat{M}_t(\epsilon^{t-1})}{\hat{P}_t(\epsilon^t)} + \hat{b}_{t+1}(\epsilon^t) - \hat{b}_t(\epsilon^{t-1})\hat{R}_t(\epsilon^t) \\ = \tau_t(\epsilon^t) + \frac{M_{t+1}(\epsilon^t) - M_t(\epsilon^{t-1})}{P_t(\epsilon^t)} + b_{t+1}(\epsilon^t) - b_t(\epsilon^{t-1})R_t(\epsilon^t), \\ \hat{P}_t(\epsilon^t) = \hat{M}_{t+1}(\epsilon^t)/w_t(\epsilon^t), \\ 0 = \mathbb{E}_t \left[m_{t+1}(\epsilon^{t+1})(\hat{R}_{t+1}(\epsilon^{t+1}) - R_{t+1}(\epsilon^{t+1})) \right]. \end{aligned}$$

We consider several special cases.

Current vs. Future Taxes Delaying taxation and financing the temporary revenue shortfall by issuing government debt leaves the money supply unchanged. The equilibrium price level sequence then is unchanged as well. For example, in a two-period setup, altering taxes and debt issuance to

$$\begin{aligned} \hat{\tau}_0(\epsilon^0) &= \tau_0(\epsilon^0) - \Delta, \\ \hat{b}_1(\epsilon^0) &= b_1(\epsilon^0) + \Delta, \\ \hat{\tau}_1(\epsilon^1) &= \tau_1(\epsilon^1) + R_1(\epsilon^1)\Delta \end{aligned}$$

has no effect on price levels.

Seignorage vs. Future Taxes A one-time change of the composition of financing between seignorage and debt coupled with a subsequent change of taxes has a permanent

effect on the price level. Formally, let

$$\begin{aligned}\hat{\tau}_0 &= \tau_0, \\ \hat{b}_1(\epsilon^0) + \frac{\hat{M}_1(\epsilon^0) - M_0}{\hat{P}_0(\epsilon^0)} &= b_1(\epsilon^0) + \frac{M_1(\epsilon^0) - M_0}{P_0(\epsilon^0)}, \\ \hat{P}_0(\epsilon^0) &= \hat{M}_1(\epsilon^0)/w_0(\epsilon^0),\end{aligned}$$

where the two seignorage terms differ. Since the policy $\hat{\varphi}$ does not involve further changes in seignorage the effect on the price level is permanent.³ The change of debt issuance at date $t = 0$ implies $\hat{b}_1(\epsilon^0)\hat{R}_1(\epsilon^1) \neq b_1(\epsilon^0)R_1(\epsilon^1)$ for some history ϵ^1 . Long-term budget balance therefore requires an appropriate adjustment of taxes subsequent to ϵ^1 . With this adjustment, all equilibrium conditions are met.

An open market operation where the government repurchases government debt from households can be viewed as an instance of this case.

Current vs. Future Seignorage A reduction of seignorage financed by debt, coupled with a subsequent increase of seignorage permanently alters the price level sequence. In fact, such a monetary contraction coupled with a subsequent expansion implies a higher price level in the long run.

To understand this “*unpleasant monetarist arithmetic*,” consider for simplicity a deterministic environment with constant endowment, w , and gross interest rate, $R > 1$. Feasible policy φ involves no seignorage revenues, $M_t = M$, such that $P_t = P = M/w$. Under the modified policy, $\hat{\varphi}$, money balances are reduced at date $t = 0$ and kept constant until date $t = T - 1$ when they are increased again. That is,

$$\begin{aligned}\hat{M}_{t+1} &= M - \Delta_1, \quad t = 0, \dots, T - 2, \\ \hat{M}_{t+1} &= M - \Delta_1 + \Delta_T, \quad t = T - 1, T, \dots\end{aligned}$$

The cash-in-advance constraints imply

$$\begin{aligned}\hat{P}_t &= (M - \Delta_1)/w, \quad t = 0, \dots, T - 2, \\ \hat{P}_t &= (M - \Delta_1 + \Delta_T)/w, \quad t = T - 1, T, \dots\end{aligned}$$

Under $\hat{\varphi}$, seignorage revenues at date $t = 0$ and $t = T - 1$, respectively, equal $-\Delta_1 w / (M - \Delta_1)$ and $\Delta_T w / (M - \Delta_1 + \Delta_T)$. To satisfy (11.2), the present values of these revenues must equal zero, implying

$$\frac{\Delta_T w}{M - \Delta_1 + \Delta_T} = R^{T-1} \frac{\Delta_1 w}{M - \Delta_1} \Rightarrow \frac{\Delta_T}{\Delta_1} = R^{T-1} \frac{M - \Delta_1 + \Delta_T}{M - \Delta_1}.$$

Note that $\Delta_T > \Delta_1$, both since $R > 1$ and $\hat{P}_{T-1} > \hat{P}_0$. That is, money balances and thus, the price level increase in the long run as a consequence of a monetary contraction

³To see this, consider for simplicity a deterministic setting. The cash-in-advance constraints imply $\{\hat{M}_{t+1}/\hat{P}_t\}_{t \geq 0} = \{M_{t+1}/P_t\}_{t \geq 0}$. $(\hat{M}_1 - M_0)/\hat{P}_0 \neq (M_1 - M_0)/P_0$ implies $\hat{P}_0 \neq P_0$ and $\hat{M}_1 \neq M_1$. Unchanged seignorage revenues in period $t = 1$, $(\hat{M}_2 - \hat{M}_1)/\hat{P}_1 = (M_2 - M_1)/P_1$, implies $\hat{M}_1/\hat{P}_1 = M_1/P_1$ and thus, $\hat{P}_1 \neq P_1$ and $\hat{M}_2 \neq M_2$. The argument extends to subsequent periods.

at date $t = 0$. While inflation reflects money growth, monetary policy is forced to generate inflation for fiscal reasons. Note also that a delay of the compensating policy change increases the long-run price level (P_T increases in T): Postponing the necessary inflation to generate seignorage increases the inflation.

11.4.5 Game Of Chicken

When monetary and fiscal policy are controlled by separate authorities—a central bank on the one hand and a fiscal authority on the other—then the institutional structure governing their interplay can have important macroeconomic implications.

Suppose the fiscal authority “moves first” in the sense of committing to state-contingent tax and government consumption sequences before the monetary authority chooses the money supply and thus, the seignorage revenue to contribute to the government budget. By moving first, the fiscal authority shifts responsibility for implementing an equilibrium to the monetary authority; the latter must generate sufficient seignorage to satisfy the intertemporal budget constraint. In this “*game of chicken*” the central bank’s choice set thus is restricted by the actions of the fiscal authority. Although the central bank may wish to conduct a monetary policy aimed at stabilizing the price level say, its second mover status can render that impossible.

Threats to price stability of this kind can be countered by instituting a different arrangement that guarantees *central bank independence* and assigns the first mover advantage to the monetary authority. An independent central bank is relieved of the responsibility for intertemporally balancing the budget; that responsibility lies with the fiscal authority.

11.4.6 Fiscal Theory of the Price Level

When nominal in addition to real debt is outstanding at date $t = 0$, the government’s intertemporal budget constraint (11.2) may not only be satisfied by appropriately choosing government consumption, taxes or seignorage revenues, but also by revaluing the stock of nominal debt through a change of price level in the initial period. The “*fiscal theory of the price level*” (FTPL) emphasizes this possibility. It views the intertemporal budget constraint (11.2) not as a constraint on government actions but only as an equilibrium condition.

The FTPL can be motivated in a one-period model. Suppose that nominal liabilities B_0I_0 are outstanding at date $t = 0$ which is the last period, and money balances and seignorage both are negligible. Suppose further that the choice of fiscal policy is “*non-Ricardian*.” Rather than balancing the government’s budget by setting $\tau_0 = B_0I_0/P_0$ for whatever equilibrium price level is realized (as would be the case in the “*Ricardian*” case), the fiscal authority sets τ_0 independently of P_0 . An equilibrium then may still exist as long as $P_0 = B_0I_0/\tau_0$. In this case, the price level is fiscally determined.⁴

⁴There is an even simpler mechanism without debt to fiscally determine the price level: When the fiscal authority sets government consumption in nominal terms, G_0 , and real tax revenue, τ_0 , then budget

For a more careful analysis, consider the model introduced in subsection 11.4.3. We abstract from real debt and assume that the nominal interest rate is non-state-contingent such that $B_{t+1}(\epsilon^t)I_{t+1}(\epsilon^t)$ is constant for all $\epsilon^{t+1}|\epsilon^t$. Accordingly, the dynamic budget constraint of the government reads

$$\frac{B_t(\epsilon^{t-1})I_t(\epsilon^{t-1})}{P_t(\epsilon^t)} = \tau_t(\epsilon^t) - g_t(\epsilon^t) + \frac{B_{t+1}(\epsilon^t)}{P_t(\epsilon^t)} + \frac{M_{t+1}(\epsilon^t) - M_t(\epsilon^{t-1})}{P_t(\epsilon^t)},$$

and the intertemporal budget constraint at date $t = 0$ is given by

$$\frac{B_0(\epsilon^{-1})I_0(\epsilon^{-1}) + M_0(\epsilon^{-1})}{P_0(\epsilon^0)} = \sum_{j=0}^{\infty} \mathbb{E}_0 \left[(m_1(\epsilon^1) \cdots m_j(\epsilon^j)) \left(\tau_j(\epsilon^j) - g_j(\epsilon^j) + \frac{m_{j+1}(\epsilon^{j+1}) M_{j+1}(\epsilon^j) i_{j+1}(\epsilon^{j+1})}{P_{j+1}(\epsilon^{j+1})} \right) \right].$$

The remaining equilibrium conditions are

$$\begin{aligned} 1 &= \mathbb{E}_t \left[m_{t+1}(\epsilon^{t+1}) I_{t+1}(\epsilon^t) \Pi_{t+1}^{-1}(\epsilon^{t+1}) \right], \\ P_t(\epsilon^t) &= \frac{M_{t+1}(\epsilon^t)}{w_t(\epsilon^t)}, \end{aligned}$$

where $m_{t+1}(\epsilon^{t+1})$ is pinned down by the resource constraint.

A *policy regime* is a mapping from a set \mathcal{P} of positive price level sequences into a set whose elements are sets of policies. With each price level sequence $\{P_t(\epsilon^t)\}_{t \geq 0}$ in \mathcal{P} , the policy regime associates a set of policies, $\Phi(\{P_t\}_{t \geq 0})$, such that the price level sequence and any policy in the set satisfy the above equilibrium conditions, except possibly the intertemporal budget constraint. A policy regime is Ricardian if the price level sequence and policies also satisfy the intertemporal budget constraint. Otherwise, the policy regime is non-Ricardian: there exist some price level sequences in \mathcal{P} and associated policies that do not satisfy the intertemporal budget constraint. These price level sequences then are ruled out as equilibrium price level sequences.

Consider for simplicity a deterministic environment with constant endowments, $w_t = w$, and government consumption, $g_t = g$, implying $m_{t+1} = \beta$. A (Ricardian or non-Ricardian) policy regime imposes the following restrictions on policy:

$$\begin{aligned} M_{t+1} &= P_t w, \\ I_{t+1} &= \Pi_{t+1} / \beta, \\ B_{t+1} &= B_t I_t - P_t \left(\tau_t - g + w - \Pi_t^{-1} w \right). \end{aligned}$$

If the regime is Ricardian then it also imposes the intertemporal budget constraint which, using the previous conditions, reduces to

$$\frac{B_0 I_0 + M_0}{P_0} = \sum_{j=0}^{\infty} \beta^j \left(\tau_j - g + \frac{w i_{j+1}}{I_{j+1}} \right),$$

balance implies $P_0 = G_0 / \tau_0$.

where $i_t \equiv I_t - 1$. If the regime is non-Ricardian, in contrast, then it does not impose the latter restriction on policy. A non-Ricardian regime therefore may rule out certain price level sequences that are not ruled out under a Ricardian regime. In that sense, the non-Ricardian policy regime determines the price level (sequence).

Suppose that $B_0 I_0 + M_0 \neq 0$ and monetary policy fixes the nominal interest rate at value I . Equilibrium inflation then is constant at value $\Pi = \beta I$ and money supply grows at the gross rate Π , implying $P_t = P_0 \Pi^t$ and $M_{t+1} = P_0 \Pi^t w$. We consider fiscal policy regimes that relate positive price level sequences growing at a constant rate, to fiscal policies $\{\tau_t, g, B_{t+1}\}_{t \geq 0}$. In any fiscal policy regime, fiscal policy satisfies

$$B_{t+1} = B_t I - P_0 \Pi^t \left(\tau_t - g + w - \Pi^{-1} w \right).$$

If the fiscal policy regime is Ricardian then policy also satisfies

$$\frac{B_0 I_0 + M_0}{P_0} = \sum_{j=0}^{\infty} \beta^j \left(\tau_j - g + w \frac{I-1}{I} \right).$$

Since the latter condition constitutes a restriction on fiscal policy conditional on P_0 —fiscal policy “adjusts” to P_0 —we conclude that under a Ricardian fiscal policy regime the initial price level is indeterminate. This exemplifies a classical result in monetary economics: An interest rate peg alone leaves the price level indeterminate. Under a non-Ricardian policy regime, however, fiscal policy does not “adjust” to P_0 and the price level therefore is determinate.

Suppose alternatively that monetary policy fixes the money supply at date t at value M_{t+1} such that the equilibrium price level equals $P_t = M_{t+1}/w$. Under a non-Ricardian fiscal policy regime the price level now is over determined; except for knife edge cases, only a Ricardian fiscal policy regime is consistent with equilibrium.

That a non-Ricardian policy regime may determine the “initial” price level does not mean that the government can choose primary surpluses (including seignorage revenues) arbitrarily. Standard asset pricing implies that rational investors value nominal government liabilities at the expected present discounted value of future primary surpluses, net of the returns on outstanding real debt. When nominal debt is issued for the first time the government therefore cannot raise more resources, in present discounted value terms, than it is expected to repay to bond holders in the future. Accordingly, the intertemporal budget constraint binds at the time of debt issuance. This can also be seen by noting that, at the “truly initial” date $t = -1$ say when $B_{-1} I_{-1} + M_{-1} = 0$ the price level P_{-1} cannot revalue outstanding liabilities. A non-Ricardian policy regime therefore does not allow the government to escape long-run budget balance; it may only contribute to determining state-contingent inflation rates. Similarly, a non-Ricardian policy regime does not provide a nominal anchor before nominal liabilities have been issued for the first time.

11.4.7 Stability under Policy Rules

Mechanically, what determines the equilibrium price level under a non-Ricardian policy regime is the fact that only a specific price level prevents explosive debt dynamics.

The equilibrium conditions without the intertemporal budget constraint determine the path of government debt in real terms, conditional on its starting value, and this path satisfies the government's no-Ponzi-game condition (or the household's transversality condition) only for a specific starting value and thus, a specific initial price level.

The same mechanism is at play when a policy regime prescribes certain ad-hoc policy rules, for example rules specifying how the interest rate and taxes are set in response to inflation and the stock of outstanding debt. To understand the essence of the argument which relates to the stability of a difference equation, it suffices to consider a deterministic setting. Suppose the policy regime prescribes that the nominal interest rate responds to inflation and taxes net of government consumption respond to the stock of real debt at the end of the previous period,

$$\begin{aligned} I_{t+1} &= \alpha \Pi_t, \\ \tau_t - g_t &= \gamma \frac{B_t}{P_{t-1}}, \end{aligned}$$

where α and γ are fixed parameters. (Throughout, we drop constant terms when they are irrelevant for the argument.)

Suppose as before that the equilibrium gross real interest rate equals β^{-1} . The household's Euler equation, $I_{t+1} = \Pi_{t+1}/\beta$, and the interest rate rule then imply

$$\Pi_{t+1} = \alpha \beta \Pi_t.$$

We allow for a cash-in-advance constraint or a money demand function that depends on the interest rate; in either case, $M_{t+1}/P_t = w_t \phi(I_{t+1})$ for some function ϕ . The dynamic budget constraint then can be expressed as

$$\begin{aligned} \frac{B_{t+1}}{P_t} &= \frac{B_t}{P_{t-1}} \left(\frac{I_t}{\Pi_t} - \gamma \right) - \frac{M_{t+1} - M_t}{P_t} \\ &= \frac{B_t}{P_{t-1}} (\beta^{-1} - \gamma) - \chi(w_t, w_{t-1}, \Pi_t) \end{aligned}$$

for some function χ , where the second equality uses the Euler equation, the interest rate rule and the equilibrium condition $\Pi_{t+1} = \alpha \beta \Pi_t$.

Linearizing the two dynamic equations yields a linear difference equation system in two variables, the deviation of Π_t from its steady-state value—which is not predetermined at date t —and the deviation of B_t/P_{t-1} from its steady-state value—which is predetermined at date t . The matrix determining the stability of the system is given by

$$M \equiv \begin{bmatrix} \alpha \beta & 0 \\ \xi & \beta^{-1} - \gamma \end{bmatrix}$$

for some constant ξ ; its eigenvalues equal $\alpha \beta$ and $\beta^{-1} - \gamma$.

Strengthening the requirement in the FTPL that the household satisfies its transversality condition, we restrict attention to bounded solutions of the system. If both eigenvalues of M are unstable then no such bounded solutions exist. If both eigenvalues are

stable then any initial inflation rate together with the predetermined real debt value gives rise to bounded solutions. If one of the eigenvalues is stable and the other one is not then the system is saddle-path stable and the two policy rules pin down a unique initial inflation rate conditional on the predetermined real debt.

Specifically, if $|\alpha\beta| > 1$ (“active monetary policy”), inflation in the initial period must equal a specific value to guarantee stable inflation dynamics (the difference equation for inflation is solved forward to yield a bounded solution). But in this case, debt dynamics only are bounded if $|\beta^{-1} - \gamma| < 1$ (“passive fiscal policy”). The intuition parallels the one in the case of a Ricardian fiscal policy regime in the FTPL where fiscal policy “adjusts” to the price level or inflation to satisfy the intertemporal budget constraint.

Alternatively, if $|\beta^{-1} - \gamma| > 1$ (“active fiscal policy”), inflation (and thus, the price level) in the initial period must adjust to guarantee stable debt dynamics. Stable inflation dynamics then requires $|\alpha\beta| < 1$ (“passive monetary policy”). The intuition parallels the one in the case of a non-Ricardian fiscal policy regime subject to an interest rate peg in the FTPL.

11.5 Bibliographic Notes

Baxter and King (1993) and Barro (1990) analyze tax financed government consumption or investment in the neoclassical growth model and the Ak model, respectively. Barro (1974) discusses the Ricardian equivalence proposition. Diamond (1965) studies debt in the OLG model. Auerbach, Gokhale and Kotlikoff (1994) discuss generational accounting, Breyer (1989) and Rangel (1997) analyze equivalent social security reforms, and Ball and Mankiw (2007) analyze risk sharing properties of social security systems. Gonzalez-Eiras and Niepelt (2015) state a general neutrality result. Bassetto and Kocherlakota (2004) derive the neutrality result for policy changes that involve distorting taxes. Cagan (1956) analyzes need-for-seignorage driven (hyper)inflation. The “unpleasant monetarist arithmetic” is due to Sargent and Wallace (1981). Subsection 11.4.4 follows Sargent (1987, 5.4). Leeper (1991) analyzes “active” and “passive” policy rules, and the FTPL is due to Sims (1994) and Woodford (1995), see also Aiyagari and Gertler (1985) and Kocherlakota and Phelan (1999). For critiques, see Buiter (2002) and Niepelt (2004a). Sargent and Wallace (1975) establish price level indeterminacy under an interest rate rule.

Beyond the material covered in the chapter, Wallace (1981) and Chamley and Polemarchakis (1984) derive neutrality results in economies with money as a store of value.

Chapter 12

Optimal Policy

When policy affects the equilibrium allocation then some policies are better than others. The “*Ramsey program*” is to choose the best (according to some criterion) admissible and feasible policy that is, the best policy using instruments at the government’s disposable and implementing an equilibrium. The solution to the Ramsey program—the *Ramsey policy*—implements the *Ramsey allocation*. Both admissibility and feasibility constraints typically bind: A social planner that chooses among feasible allocations always can do at least as good as a Ramsey government that chooses among feasible allocations which can be implemented as equilibrium outcomes given the set of admissible policy instruments.

We start by reviewing fundamental results on Ramsey tax policies from public finance before turning to macroeconomic applications.

12.1 The Primal and Dual Approach to Optimal Taxation

Consider a complete markets economy with N goods and H households. Firms operate a CRTS technology whose production frontier is characterized by the function $f, y_1 = f(y_2, \dots, y_N)$, where y denotes the vector of net outputs. Taking the vector of producer prices, p , as given firms maximize profits. Price taking and CRTS imply that profits equal zero. (Alternatively, firms are price takers and all profits are fully taxed away.)

Taking the vector of consumer prices, q , as given household h chooses the vector of net supplies and demands (e.g., supply of labor, demand for consumption), x^h , to maximize utility, u^h , subject to its budget constraint. The demand function of household h for good j is denoted $x_j^h(q)$, and for all goods $x^h(q)$. The household’s indirect utility function, v^h , is defined as

$$v^h(q) \equiv \max_{x^h} u^h(x^h) \text{ s.t. } q \cdot x^h = 0.$$

Let $x \equiv \sum_h x^h$ and $x(q) \equiv \sum_h x^h(q)$.

The government maximizes a social welfare function, v , which aggregates (u^1, \dots, u^H) . It has access to a technology whose production frontier is characterized by the function

$g, z_1 = g(z_2, \dots, z_N)$, where z denotes the vector of government net outputs. The government also has access to good-specific, proportional excise taxes; consumer prices therefore may differ from producer prices, $q \neq p$.

Market clearing requires $x = y + z$. When firms and households satisfy their budget constraints and all markets clear, Walras' Law implies that the government satisfies its budget constraint as well.

12.1.1 Primal Approach

Abstract from government production and assume that the government has an exogenous resource requirement for good 1, $z_1 \leq 0$; all other components of z equal zero. In a competitive equilibrium, the first-order conditions of households and firms, the budget constraints of households, the market clearing condition, and the resource constraint are satisfied:

$$\begin{aligned} \frac{u_j^h(x^h)}{u_1^h(x^h)} &= \frac{q_j}{q_1}, \\ -f_j(y_2, \dots, y_n) &= p_j/p_1, \quad j > 1, \\ q \cdot x^h &= 0, \\ x &= y + z, \\ y_1 &= f(y_2, \dots, y_N). \end{aligned}$$

Note that we can normalize $p_1 = q_1 = 1$ because the household does not have exogenous income.

From the first-order conditions of firms, the government can select any point on the production frontier by choosing p . This choice does not affect household demand because the government may choose q independently of p and there are no profits. Accordingly, the first-order conditions of firms do not constrain the government; they only determine the price vector p that supports a feasible y . We may therefore drop the first-order conditions of firms, and thus p , from the set of equilibrium conditions that constrain the Ramsey government.

Further simplifying these conditions, we eliminate consumer prices, q , by substituting the household first-order conditions in the budget constraints, arriving at

$$\begin{aligned} \frac{u_j^1(x^1)}{u_1^1(x^1)} &= \frac{u_j^h(x^h)}{u_1^h(x^h)}, \\ \sum_{j=1}^N u_j^h(x^h) x_j^h &= 0, \\ x_1 - z_1 &= f(x_2, \dots, x_N). \end{aligned}$$

The first two conditions, the *implementability constraints*, reflect equilibrium in the private sector: Marginal rates of substitution are equal across households because all

households face the same consumer prices; and each household satisfies its budget constraint with prices expressed in terms of marginal rates of substitution. The last condition imposes market clearing subject to the resource constraint.

Note that the equations above do not involve prices. The Ramsey program thus can be specified as a constrained choice of allocation; this is referred to as the *primal approach*. Note also that the constraint set is smaller than the constraint set of a social planner; the latter does not have to satisfy the implementability constraints. Finally, note that restrictions on the admissible tax instruments, for example that certain tax rates have to be equal to each other, would give rise to additional implementability constraints.

Suppose that $H = 1$ such that there is just one implementability constraint. The Ramsey program then reads

$$\max_x u^1(x) \quad \text{s.t.} \quad \sum_{j=1}^N u_j^1(x)x_j = 0, \quad x_1 - z_1 = f(x_2, \dots, x_N).$$

Letting λ and μ denote the multipliers associated with the implementability and resource constraint, respectively, the first-order conditions read

$$\begin{aligned} u_1^1(x)[1 + \lambda(1 - E^1)] &= \mu, \\ u_j^1(x)[1 + \lambda(1 - E^j)] &= -\mu f_j(x_2, \dots, x_N), \quad j > 1, \end{aligned}$$

where we define

$$E^j \equiv - \sum_{i=1}^N \frac{u_{ji}^1(x)x_i}{u_j^1(x)}$$

as the sum of the elasticities of marginal utility, $u_j^1(x)$, with respect to each of the goods. A social planner not bound by the implementability constraint ($\lambda = 0$) optimally sets all marginal rates of substitution equal to the corresponding marginal rates of transformation. The Ramsey government cannot do that (unless $z_1 = 0$ such that the implementability constraint is slack) because equilibrium and specifically, government budget balance requires distorting taxes.

Since $u_j^1(x) = \alpha^1 q_j$ where α^1 denotes the marginal utility of income, the first-order conditions can be written as

$$\alpha^1 q_j [1 + \lambda(1 - E^j)] = \mu p_j.$$

Using $p_1 = q_1 = 1$ to solve for λ and substituting yields

$$\frac{q_j - p_j}{q_j} = 1 - \frac{\alpha^1}{\mu} [1 + \lambda(1 - E^j)] = \frac{\mu - \alpha^1 E^j - E^1}{\mu (1 - E^1)}, \quad j > 1.$$

If $-E^1 \rightarrow \infty$ (completely inelastic demand for good 1) then optimal tax rates are uniform across goods 2 to N . If $-E^1 \rightarrow 0$ (completely elastic demand for good 1) then

there are no income effects on $x_j, j > 1$, and the optimal tax rates are proportional to E^j . If, moreover, the cross-partials of the utility function equal zero (independent demands) then E^j reduces to the inverse of the own-price elasticity and the standard partial equilibrium result follows: Optimal tax rates are inversely proportional to the price elasticity.¹

Suppose that $u(x) \equiv u(w(x_1, \dots, x_n), x_{n+1}, \dots, x_N)$ where the function w is homothetic. This implies that E^j is the same for all $j \leq n$ and thus, that the first-order conditions combine to

$$\frac{u_j^1(x)}{u_1^1(x)} = -f_j(x_2, \dots, x_N), \quad 1 < j \leq n.$$

Equivalently, $q_j/p_j = q_1/p_1$, that is the optimal ad valorem tax rates on goods 1 to n are *uniform* (and equal to zero since we normalized the tax rate on good 1 to be zero).

12.1.2 Dual Approach

The *dual approach* specifies the Ramsey program as a choice of prices and thus taxes, rather than allocation. We allow for government production and multiple households and let $v(q) \equiv v(v^1(q), \dots, v^H(q))$. Normalizing $p_1 = q_1 = 1$, the program reads

$$\max_{q_2, \dots, q_N, z_2, \dots, z_N} v(q) \quad \text{s.t.} \quad x_1(q) - g(z_2, \dots, z_N) = f(x_2(q) - z_2, \dots, x_N(q) - z_N)$$

and the first-order conditions are given by

$$\begin{aligned} 0 &= \lambda (f_j(x_2(q) - z_2, \dots, x_N(q) - z_N) - g_j(z_2, \dots, z_N)), \quad j > 1, \\ \frac{\partial v(q)}{\partial q_j} &= \lambda \left(\frac{\partial x_1(q)}{\partial q_j} - \sum_{i=2}^N f_i \frac{\partial x_i(q)}{\partial q_j} \right), \quad j > 1. \end{aligned}$$

The first condition represents optimality of z_j . It implies that as long as resources are scarce ($\lambda \neq 0$) *aggregate production efficiency* is optimal: The marginal rates of transformation should be equalized across private and public and more generally, across all productive sectors. Production efficiency rules out sector specific taxes on production factors, e.g., sector specific investment subsidies; taxes on goods used as intermediate inputs in certain sectors; sector specific payroll taxes; or tariffs that drive a wedge between domestic and world market producer prices. (Note that the production efficiency result also implies the uniform commodity taxation result.)

The second condition represents optimality of q_j . Using the equilibrium expression for producer prices and letting $t_j \equiv q_j - p_j$, it can be written as

$$\frac{\partial v(q)}{\partial q_j} = \lambda \sum_{i=1}^N p_i \frac{\partial x_i(q)}{\partial q_j} = -\lambda \frac{\partial}{\partial t_j} \sum_{i=1}^N t_i x_i(q).$$

¹From $u_j^1(x) = q_j \alpha^1$, we have $u_{jj}^1(x) x_j'(q_j) = \alpha^1$ and thus, $E^j = -u_{jj}^1(x) x_j / u_j^1(x) = -x_j / [x_j'(q_j) q_j]$.

(The second equality uses the budget constraint, $\sum_i p_i x_i(q) = \sum_i (q_i - t_i) x_i(q) = -\sum_i t_i x_i(q)$.) At the optimum, a price change thus affects the social welfare function proportionally to the resource cost of meeting the induced change in demand, or proportionally to the induced change in tax revenue. The latter implication would also have followed if we had maximized the social welfare function subject to the government budget constraint (for given producer prices) rather than the resource constraint.

Roy's Identity states that $\partial v^h(q)/\partial q_j = -\alpha^h x_j^h(q)$ where α^h denotes household h 's marginal utility of income. Letting $\beta^h \equiv \partial v(q)/\partial v^h \alpha^h$ denote household h 's *social marginal utility of income*, the first-order condition for prices can be expressed as

$$\sum_{h=1}^H \beta^h x_j^h(q) = \lambda \sum_{h=1}^H \left(x_j^h(q) + \sum_{i=1}^N t_i s_{ij}^h(q) - x_j^h(q) \sum_{i=1}^N t_i \frac{\partial x_i^h(q)}{\partial I} \right).$$

Here, s_{ij}^h denotes the derivative of household h 's compensated demand for good i with respect to q_j , and $\partial x_i^h(q)/\partial I$ denotes the income effect. (The decomposition uses the Slutsky equation.) Finally, letting $\gamma^h \equiv \beta^h/\lambda + \sum_{i=1}^N t_i \partial x_i^h(q)/\partial I$ denote the *social marginal valuation of income*—the marginal valuation of h 's income given that this income also generates tax revenue—the condition simplifies to the *many-person Ramsey rule*,

$$\frac{\sum_{h=1}^H (\gamma^h - 1) x_j^h(q)}{x_j(q)} = \frac{\sum_{h=1}^H \sum_{i=1}^N t_i s_{ij}^h(q)}{x_j(q)}.$$

The right-hand side of the condition constitutes a measure of the tax induced substitution effects on $x_j(q)$ (using the symmetry of the Slutsky matrix). If producer prices were constant, consumers compensated, and the derivatives of compensated demand constant, then it would equal the relative change in demand for $x_j(q)$. The left-hand side accounts for differences in the social marginal valuation of income and the expenditures shares. If γ^h is the same for all households or $x_j^h(q)/x_j(q)$ the same across goods, then the left-hand side reduces to a constant and the tax induced substitution effects are constant across goods. Otherwise, they are not.

Suppose the government can also levy a poll tax. (An excise tax on labor supply combined with a poll tax is equivalent to a linear labor income tax.) Optimality then requires

$$\sum_{h=1}^H \beta^h = \lambda \sum_{h=1}^H \sum_{i=1}^N p_i \frac{\partial x_i^h(q)}{\partial I}.$$

Since

$$\sum_{h=1}^H \sum_{i=1}^N p_i \frac{\partial x_i^h(q)}{\partial I} = \sum_{h=1}^H \sum_{i=1}^N (q_i - t_i) \frac{\partial x_i^h(q)}{\partial I} = \sum_{h=1}^H \left(1 - \sum_{i=1}^N t_i \frac{\partial x_i^h(q)}{\partial I} \right),$$

we conclude that with a poll tax, the average γ^h equals unity. Accordingly, the left-hand side of the many-person Ramsey rule is the covariance between γ^h and $x_j^h(q)/x_j(q)$. Under the Ramsey policy, the relative change in demand for good j is proportional to

this covariance: Consumption of goods in strong demand by households with high social marginal valuation of income should be discouraged less.

With just one consumer, $v = v^1$ and the many-person Ramsey rule reduces to the condition

$$\gamma^1 - 1 = \frac{\sum_{i=1}^N t_i s_{ij}^1(q)}{x_j(q)},$$

implying that the tax induced substitution effects should be constant across goods. If a poll tax is available such that $\gamma^1 = 1$ no commodity taxes are levied.

12.2 Tax Smoothing

In the macroeconomic context, the public finance question of *which* goods to tax corresponds to the problem of *when* to tax in order to minimize tax distortions or negative distributive implications. From an efficiency viewpoint optimal tax rates typically fluctuate little over time and across states that is, government spending and taxation are decoupled. The key instrument to sustain such decoupling is government indebtedness and insurance.

12.2.1 Complete Markets

Consider a representative household economy without capital. The household is endowed with one unit of time per period. At date t , history ϵ^t time can be transformed into $w_t(\epsilon^t)$ units of the good. Household preferences over consumption, c , and leisure, x , are represented by the standard utility function $\mathbb{E}_0[\sum_{t=0}^{\infty} \beta^t u(c_t(\epsilon^t), x_t(\epsilon^t))]$ where u is strictly concave and β denotes the discount factor.

To finance a given stream of government consumption, $\{g_t(\epsilon^t)\}_{t \geq 0}$, the government taxes labor income at rates $\{\tau_t(\epsilon^t)\}_{t \geq 0}$ and issues Arrow-Debreu securities of arbitrary maturity; markets are complete. Without loss of generality, taxes on consumption are normalized to zero (see the discussion in subsection 11.3.2 and note that absent a technology to transform resources intertemporally, there are no intertemporal producer prices).

Variable ${}_t b_s(\epsilon^{t-1}, \epsilon^s)$, $s \geq t$, denotes claims vis-a-vis the government held at date t , given that history ϵ^{t-1} occurred; one claim entitles to one unit of the consumption good at date s after history ϵ^s conditional on history ϵ^{t-1} . The marginal distribution of ϵ^t is denoted by $F_t(\epsilon^t)$ and its density by $f_t(\epsilon^t)$.

We adopt the primal approach. The benevolent government maximizes household welfare subject to the resource constraints

$$c_t(\epsilon^t) + g_t(\epsilon^t) = w_t(\epsilon^t)(1 - x_t(\epsilon^t))$$

and the implementability constraint

$$\sum_{t=0}^{\infty} \int \beta^t [u_c(c_t(\epsilon^t), x_t(\epsilon^t))(c_t(\epsilon^t) - {}_0 b_t(\epsilon^{-1}, \epsilon^t)) - u_x(c_t(\epsilon^t), x_t(\epsilon^t))(1 - x_t(\epsilon^t))] dF_t(\epsilon^t) = 0.$$

The latter integrates the household's (complete markets) intertemporal budget constraint,

$$\sum_{t=0}^{\infty} \int q_t(\epsilon^t) [c_t(\epsilon^t) - (1 - \tau_t(\epsilon^t))w_t(\epsilon^t)(1 - x_t(\epsilon^t)) - {}_0b_t(\epsilon^{-1}, \epsilon^t)] d\epsilon^t = 0,$$

as well as the household first-order conditions,

$$\begin{aligned} u_c(c_0, x_0)q_t(\epsilon^t) &= \beta^t f_t(\epsilon^t) u_c(c_t(\epsilon^t), x_t(\epsilon^t)), \\ u_c(c_t(\epsilon^t), x_t(\epsilon^t))w_t(\epsilon^t)(1 - \tau_t(\epsilon^t)) &= u_x(c_t(\epsilon^t), x_t(\epsilon^t)), \end{aligned}$$

where $q_t(\epsilon^t)$ denotes the price at date $t = 0$ of a unit of the good at date t after history ϵ^t . (The resource and budget constraints imply the government's intertemporal budget constraint.)

The government's choice variables are consumption and leisure. Let ν and $-\beta^t \mu_t(\epsilon^t) f_t(\epsilon^t)$ denote the multipliers associated with the implementability and resource constraints, respectively. Suppressing histories to improve legibility, the first-order conditions are given by

$$\begin{aligned} (1 + \nu)u_c(c_t, x_t) + \nu(u_{cc}(c_t, x_t)(c_t - {}_0b_t) - u_{xc}(c_t, x_t)(1 - x_t)) &= \mu_t, \\ (1 + \nu)u_x(c_t, x_t) + \nu(u_{cx}(c_t, x_t)(c_t - {}_0b_t) - u_{xx}(c_t, x_t)(1 - x_t)) &= \mu_t w_t. \end{aligned}$$

The conditions state that the government accounts for three types of effects when increasing c_t or x_t . First, the direct effects on the objective function. Second, the resource costs, represented by the terms multiplying μ_t . And third, the marginal effects on the implementability constraint, represented by the terms multiplying ν . The latter effects reflect both higher outlays for consumption or leisure and changes of the marginal rates of substitution—corresponding to changed inter- and intratemporal prices.

Together with the implementability and resource constraints the first-order conditions fully characterize the Ramsey allocation. From the household's first-order conditions, the latter implies a history contingent sequence of optimal tax rates. Finally, from the government's intertemporal budget constraint, the taxes imply a unique sequence of optimal government indebtedness since the value of outstanding debt equals the market value of future primary surpluses.

We make three key observations. First, in stochastic environments, the optimal indebtedness implied by the Ramsey tax sequence generally is stochastic. That is, the rate of return on government debt is not risk-free. Second and related, the *shadow value of public funds*—the government's valuation of public relative to private sector wealth, ν —is constant over time and across histories. And third, tax rates at date t only depend on $({}_0b_t, w_t, g_t)$.

To understand the first and second point, recall that the implementability constraint incorporates all competitive equilibrium conditions beyond the resource constraint. The multiplier associated with the implementability constraint therefore gives the shadow cost of the competitive equilibrium requirement; specifically, it represents the cost of the fact that taxes distorts the allocation. With complete markets households

smooth the shadow value of income over time and across histories, and a parallel result holds for the government. As a consequence, complete markets imply that the difference between the shadow value of government and private sector funds, ν , is constant.

If, in contrast, the government did not face complete markets but, to take an extreme example, had to satisfy a balanced budget restriction at each date and history then the single implementability constraint would be replaced by a history contingent sequence of such constraints with an associated sequence of multipliers. In that case, the government's inability to decouple the timing of tax collections on the one hand and government spending on the other would imply that the shadow cost of public funds varies over time and across histories. We return to this point in subsection 12.2.2.

To understand the third point on the structure of optimal taxes, note that the two first-order conditions combine to an equation in $(\nu, {}_0b_t, w_t, c_t, x_t)$ while the resource constraint contains the variables (g_t, w_t, c_t, x_t) . Since the structure of either equation is history independent the equilibrium allocation at a date and history is an invariant function of the exogenous state, $({}_0b_t, w_t, g_t)$, as well as of the constant multiplier, ν . As a consequence, the tax rates in two histories are identical as well as long as the contemporaneous state is the same in the two histories. In environments with additional state variables, e.g. capital, this complete markets result generalizes.

To characterize the Ramsey tax policy in more detail, we manipulate the optimality conditions to find two auxiliary conditions,

$$\begin{aligned} (1 + \nu)[u_c(c_t, x_t)(c_t - {}_0b_t) - u_x(c_t, x_t)(1 - x_t)] + \nu Q_t + (g_t + {}_0b_t)\mu_t &= 0, \\ \nu Q + \sum_{t=0}^{\infty} \int \beta^t (g_t + {}_0b_t)\mu_t dF_t &= 0, \end{aligned}$$

where $Q_t < 0, Q < 0$, and $\mu_t > 0$.² The first condition implies that, absent government spending in a history ($g_t = {}_0b_t = 0$), the tax rate nevertheless is strictly positive when public funds are scarce ($\nu > 0$). The second condition states that the shadow value of public funds equals zero if the market value of government consumption and predetermined debt service equals zero.

To understand the implications of these findings, suppose first that $g_t + {}_0b_t = 0$ in all histories or that $\sum_{t=0}^{\infty} \int \beta^t u_c(c_t, x_t)(g_t + {}_0b_t)dF_t = 0$. As we have just seen, $\nu = 0$ in this case. The first-order conditions then imply that the allocation is not distorted, $u_c(c_t, x_t)w_t = u_x(c_t, x_t)$, and tax rates therefore equal zero. Intuitively, there is no point in levying distorting taxes when the government's income from initial asset holdings (negative ${}_0b_t$) suffices to finance government consumption.

When $\nu > 0$, in contrast, then the government needs to raise taxes. Suppose next that $(w_t, g_t, {}_0b_t)$ is constant across histories. As discussed above, (c_t, x_t) and thus, tax rates then are constant as well. Accordingly, the government budget is balanced at all times.

²The first equation results from multiplying the government's first-order conditions by $c_t - {}_0b_t$ and $x_t - 1$, respectively, summing them, and using the resource constraint. The second equation follows from integrating the first condition, weighting by β^t , summing over time, and using the intertemporal budget constraint.

Third, consider a deterministic environment with constant productivity, w , and $g_t = 0$ at all dates except at date $t = T$ when $g_T > 0$. (From now on, we let ${}_0b_t = 0$ in all histories.) Our findings imply that tax rates at all dates $t \neq T$ are constant and strictly positive. Intuitively, since tax distortions are convex in the tax rate, optimal tax rates vary less than government spending—the optimal *tax smoothing* policy spreads tax collections over time to reduce average tax distortions. Accordingly, the government accumulates assets before date $t = T$ and services debt thereafter.

Next, consider the same scenario except that at date $t = T$, government consumption is stochastic and can take two values: $g_T > 0$ or $g_T = 0$. Our findings imply that tax rates are constant and strictly positive except at date $t = T$ when $g_T > 0$. Intuitively, the tax smoothing prescription applies both over time and across histories. Implementing an equilibrium with constant tax revenue before and after date $t = T$ requires the government's indebtedness at date $t = T + 1$ to be independent of g_T . Since the government budget at date $t = T$ is not balanced this requires that the government's indebtedness at date $t = T$ is contingent: When $g_T > 0$ then government debt is lower than when $g_T = 0$. That is, between $t = T - 1$ and $t = T$, the rate of return on government debt is contingent on the realization of g_T —the private sector (partially) insures the government against the high government consumption shock.

Finally, if (w_t, g_t) follows a deterministic cycle then (c_t, x_t) and tax rates follow a deterministic cycle as well and the government's budget is balanced over the cycle. Similarly, if (w_t, g_t) follows a stationary Markov process then (c_t, x_t) and tax rates inherit the stochastic properties of the state.

We have seen that with complete markets and stochastic (w_t, g_t) , the tax smoothing Ramsey policy relies on contingent government indebtedness. One mechanism to deliver such contingency is to make the coupon payment contingent on the realization of the state. A more subtle mechanism, which works even when coupons are risk-free, relies on an appropriate choice of maturity structure. Since shocks to productivity and government consumption alter equilibrium consumption they also affect the term structure of interest rates which in turn affects the market value of outstanding long-term debt. For example, a rise in interest rates devalues outstanding long-term debt while it does not devalue maturing liabilities. Generically, the contingent government indebtedness under the complete markets Ramsey policy is spanned by the contingent term structure of interest rates associated with the Ramsey allocation. That is, even with risk-free coupons on government debt, markets are complete as long as the maturity structure of government debt is sufficiently rich.

12.2.2 Incomplete Markets

Short-Term, Risk-Free Debt Assume now that the government only issues one-period debt, with a risk-free coupon, implying that government indebtedness is non-contingent. The complete markets Ramsey allocation characterized in subsection 12.2.1 may no longer be implementable in this case and the properties of the Ramsey policy change.

Let $b_t(\epsilon^{t-1})$ denote debt with a safe return at date t that is, claims vis-a-vis the government that are due at date t in any history subsequent to the specific history ϵ^{t-1}

(the claims are measurable with respect to ϵ^{t-1} rather than ϵ^t as before). For convenience, we assume that productivity equals unity at all times. Using the resource constraint and adopting a shorthand notation, let $u_t(\epsilon^t) \equiv u(c_t(\epsilon^t), 1 - c_t(\epsilon^t) - g_t(\epsilon^t))$, $u_{c,t}(\epsilon^t) \equiv u_c(c_t(\epsilon^t), 1 - c_t(\epsilon^t) - g_t(\epsilon^t))$, and similarly for $u_{x,t}(\epsilon^t)$.

In competitive equilibrium, the household satisfies its intratemporal first-order condition and stochastic Euler equation (due to market incompleteness); the household or equivalently, the government satisfies its dynamic budget constraint; government debt or assets are bounded; and the resource constraint is met. From the intratemporal first-order condition and the resource constraint, we can express the government's primary surplus, $\tau_t(\epsilon^t)(1 - x_t(\epsilon^t)) - g_t(\epsilon^t)$, as

$$s_t(\epsilon^t) \equiv \left(1 - \frac{u_{x,t}(\epsilon^t)}{u_{c,t}(\epsilon^t)}\right) (c_t(\epsilon^t) + g_t(\epsilon^t)) - g_t(\epsilon^t).$$

Accordingly, the government's dynamic budget constraint incorporating the household optimality conditions and the resource constraint reads

$$b_t(\epsilon^{t-1}) \leq s_t(\epsilon^t) + \beta \mathbb{E}_t \left[\frac{u_{c,t+1}(\epsilon^{t+1})}{u_{c,t}(\epsilon^t)} b_{t+1}(\epsilon^t) \right],$$

where we assume that the government may pay lump-sum transfers, thus the inequality constraint.

Iterating this equation (with equality) forward, applying the law of iterated expectations, and assuming $\lim_{T \rightarrow \infty} \beta^T u_{c,T}(\epsilon^T) = 0$ almost surely, yields the implementability constraint

$$u_{c,0} b_0 = \sum_{t=0}^{\infty} \int \beta^t u_{c,t} s_t dF_t(\epsilon^t),$$

where we suppress histories to improve legibility. (We will go back and forth between suppressing histories or not.) There are two differences between this implementability constraint and the one in subsection 12.2.1. First, the constraint here incorporates the resource constraint. Second, it is derived from the government's rather than the private sector's intertemporal budget constraint. Neither difference is important.

However, the implementability constraint does not yet reflect the restriction that indebtedness be non-contingent (see also the discussion of equation (4.1) on page 47). To incorporate this restriction, we additionally impose the condition that debt at date t and thus, the present value of primary surpluses from date t onwards be the same for all ϵ^t conditional on ϵ^{t-1} . This *measurability constraint* can be stated as

$$u_{c,t} b_t = \sum_{s=t}^{\infty} \int \beta^{s-t} u_{c,s} s_s dF_s(\epsilon^s | \epsilon^t) \forall \epsilon^t | \epsilon^{t-1}, t \geq 1,$$

where b_t on the left-hand side of the condition is measurable with respect to ϵ^{t-1} . Note that the measurability constraint at date $t \geq 1$ has the same form as the implementability constraint that holds at date $t = 0$. We also impose boundedness conditions, requiring that b_t and thus, the right-hand side of the measurability constraint normalized by $u_{c,t}$, lies between some bounds \underline{M} and \overline{M} .

Let $\beta^t f_t(\epsilon^t) \gamma_t(\epsilon^t)$ denote the multiplier associated with the implementability constraint (for $t = 0$) and the measurability constraints (for $t \geq 1$) with $\gamma_0 \leq 0$; and let $\beta^t f_t(\epsilon^t) \xi_{1,t}(\epsilon^t)$ and $\beta^t f_t(\epsilon^t) \xi_{2,t}(\epsilon^t)$ denote the multipliers associated with the upper and lower bounds, respectively. The Lagrangian of the government's program reads

$$\mathcal{L} = \sum_{t=0}^{\infty} \int \beta^t \left\{ u_t + u_{c,t}(\gamma_t b_t + \xi_{1,t} \bar{M} - \xi_{2,t} \underline{M}) - (\gamma_t + \xi_{1,t} - \xi_{2,t}) \left(\sum_{s=t}^{\infty} \int \beta^{s-t} u_{c,s} s_s dF_s(\epsilon^s | \epsilon^t) \right) \right\} dF_t(\epsilon^t).$$

This can be rewritten as

$$\mathcal{L} = \sum_{t=0}^{\infty} \int \beta^t \left\{ u_t + u_{c,t}(\gamma_t b_t + \xi_{1,t} \bar{M} - \xi_{2,t} \underline{M}) - u_{c,t} s_t \sum_{s=0}^t (\gamma_s + \xi_{1,s} - \xi_{2,s}) \right\} dF_t(\epsilon^t)$$

or

$$\begin{aligned} \mathcal{L} &= \sum_{t=0}^{\infty} \int \beta^t \left\{ u_t + u_{c,t}(\gamma_t b_t + \xi_{1,t} \bar{M} - \xi_{2,t} \underline{M}) - u_{c,t} s_t v_t \right\} dF_t(\epsilon^t) \\ \text{s.t.} \quad v_t &= v_{t-1} + \gamma_t + \xi_{1,t} - \xi_{2,t}, \quad v_{-1} = 0. \end{aligned}$$

Differentiating with respect to $c_t(\epsilon^t)$ and $b_{t+1}(\epsilon^t)$ yields the first-order conditions

$$\begin{aligned} u_{c,t} - u_{x,t} - v_t((u_{cc,t} - u_{cx,t})s_t + u_{c,t} s_{c,t}) + (u_{cc,t} - u_{cx,t})(\gamma_t b_t + \xi_{1,t} \bar{M} - \xi_{2,t} \underline{M}) &= 0, \\ \int \gamma_{t+1} u_{c,t+1} dF_{t+1}(\epsilon^{t+1} | \epsilon^t) &= 0, \end{aligned}$$

respectively. The first optimality condition relates the allocation in a period to the level of debt as well as to multipliers which may not only vary over time ($\gamma_t, \xi_{1,t}, \xi_{2,t}$) but also accumulate (v_t). The second condition states that the shadow cost of the measurability constraint in utility terms, $\gamma_{t+1} u_{c,t+1}$, should equal zero on average.

To understand these conditions, consider first the hypothetical complete-markets case where debt service at date t is measurable with respect to ϵ^t rather than ϵ^{t-1} (and $\xi_{1,t} = \xi_{2,t} = 0$). The second optimality condition then changes to $\gamma_{t+1} = 0$ and v_t is constant across histories; the optimality conditions thus reduce to the equivalent of the conditions in subsection 12.2.1. Intuitively, with complete markets, there is no cost associated with having to satisfy the intertemporal budget constraint after the initial period, conditional on satisfying it in the initial period. Stated differently, the optimal choice of contingent indebtedness equalizes the shadow cost of the government budget constraint across histories.

With incomplete markets, in contrast, the government cannot equalize the shadow cost across histories. With a risk-free debt position it can only equalize this cost on average, over time. After a negative shock to the budget the intertemporal budget constraint tightens ($\gamma_t < 0$ and v_t decreases) and going forward, the Ramsey allocation is more distorted than it would have been after a positive shock which leads to

an increase of v_t . The tightening or relaxation of the budget constraint is permanently reflected in the multiplier v and thus, in the Ramsey allocation and tax policy. In contrast to the complete markets case, tax rates therefore do not only inherit the stochastic properties of the government consumption shock but also reflect its history.

It is instructive to consider the special case with linear utility, $u(c, x) = c + H(x)$ where H is increasing and concave. From the household's first-order conditions, the asset pricing kernel then equals β , and $\tau_t = 1 - H'(x)$. Tax revenue thus is a function of x_t ,

$$\rho(x_t) = (1 - H'(x_t))(1 - x_t)$$

say. Under standard assumptions, it is a strictly concave function over the domain $[\underline{x}, \bar{x}]$ where \underline{x} denotes the undistorted level of leisure ($\rho(\underline{x}) = 0$) and $\bar{x} < 1$ denotes the level where $\rho(\bar{x})$ attains the maximum of the Laffer curve. Inverting ρ yields leisure as a strictly convex function of tax revenue, $\chi(\rho_t)$ say. Function χ is defined over the domain $[0, \rho(\bar{x})]$.

Since utility at date t equals $1 - \chi(\rho_t) - g_t + H(\chi(\rho_t))$ we may formulate the Ramsey program with tax revenue and debt as the choice variables. The program reads

$$\begin{aligned} \max_{\{\rho_t(\epsilon^t), b_{t+1}(\epsilon^t)\}_{t \geq 0}} & - \sum_{t=0}^{\infty} \int \beta^t \mathcal{D}(\rho_t) dF_t(\epsilon^t) \\ \text{s.t.} & b_t \leq \rho_t - g_t + \beta b_{t+1}, \\ & \underline{M} \leq b_{t+1} \leq \bar{M}, \end{aligned}$$

where we define the *deadweight loss* $\mathcal{D}(\rho_t) \equiv \chi(\rho_t) - H(\chi(\rho_t))$. Note that \mathcal{D} is strictly convex over the domain $[0, \rho(\bar{x})]$ and reaches a minimum at $\rho_t = 0$.

This Ramsey program is isomorphic to a consumption-saving problem with the utility function $-\mathcal{D}(\rho_t)$, which has a bliss point at $\rho_t = 0$ (no deadweight loss); an interest rate equal to the inverse of the time discount factor; negative income shocks (government consumption); an asset with a risk-free return; and the natural borrowing limit. If the Markov process for government consumption has a nontrivial invariant distribution then "utility" in this program converges to the bliss point that is, the Ramsey tax rate converges to zero. The government accumulates a sufficiently large stock of assets to finance an infinite sequence of maximum government consumption; whenever the realization of government consumption is lower than its maximum value the government pays lump-sum transfers to the households.

This can also be seen from the optimality conditions derived earlier. Under the linear utility assumption these conditions are given by

$$\begin{aligned} 1 - H'(x_t) - v_t[1 - H'(x_t) + (1 - x_t)H''(x_t)] &= 0, \\ v_t = v_{t-1} + \gamma_t + \xi_{1,t} - \xi_{2,t}, v_{-1} &= 0, \\ \mathbb{E}_t[\gamma_{t+1}] &= 0, \end{aligned}$$

where the first condition can be expressed as $-\mathcal{D}'(\rho_t) = v_t$. Whenever the debt limits do not bind v_t and thus, the marginal deadweight loss follows a martingale. That is, the Ramsey policy keeps the expected marginal tax distortion constant over time.

Compare this to a complete markets environment where the Ramsey policy keeps the marginal tax distortion constant across all histories.

Since the government can pay lump-sum transfers, $\zeta_{2,t} = 0$, v_t is nonpositive, and $\mathbb{E}_t[v_{t+1}] = v_t + \mathbb{E}_t[\zeta_{1,t+1}] \geq v_t$ that is, v_t is a nonpositive submartingale. If the process for government consumption has an absorbing state then v_t and taxes converge to a strictly negative and positive value, respectively. Otherwise, v_t and taxes converge to zero as the government accumulates assets. With an ad-hoc restriction on the accumulation of government assets, $\zeta_{2,t}$ may differ from zero; this undermines the convergence result.

A Broader Portfolio Returning to the case with general preferences, assume next that the government holds a broader portfolio of liabilities and assets, including physical capital. Markets are incomplete.

We assume that a Markov process governs government consumption and productivity, and we formulate the Ramsey program recursively. Output $f(k_o, 1 - x_o(\epsilon_o), \epsilon_o)$ depends on the predetermined capital stock, k_o , labor input, $1 - x_o(\epsilon_o)$, and a productivity shock, reflected by ϵ_o . To simplify the notation we let $u_c(\epsilon_o) \equiv u_c(c_o(\epsilon_o), x_o(\epsilon_o))$, $f_K(\epsilon_o) \equiv f_K(k_o, 1 - x_o(\epsilon_o), \epsilon_o)$, etc.

The state at the beginning of a period, *before* the realization of the shock, includes the economy's capital stock, k_o ; the government's net liabilities, b_o ; the shock in the previous period, ϵ_- ; and marginal utility in the previous period, θ_- . The choice variables in the government's program include the risk-free interest rate on government debt, R_o ; government holdings of capital, k_o^g ; exposures to arbitrary securities (in zero net supply), $\{e_o^i\}_i$, with exogenous gross returns $\{R^i(\epsilon_o)\}_i$; as well as variables which vary with the shock realization, namely consumption and leisure, $c_o(\epsilon_o)$ and $x_o(\epsilon_o)$; the capital stock at the beginning of the subsequent period, $k_+(\epsilon_o)$; government net liabilities at the beginning of the subsequent period, $b_+(\epsilon_o)$; and the labor income tax rate, $\tau_o(\epsilon_o)$. (In the initial period, the risk-free interest rate is given.)

The constraints of the government's program are given by

$$\begin{aligned}\theta_- &= \beta \mathbb{E}[u_c(\epsilon_o) R_o | \epsilon_-], \\ \theta_- &= \beta \mathbb{E}[u_c(\epsilon_o) (1 + f_K(\epsilon_o) - \delta) | \epsilon_-], \\ \theta_- &= \beta \mathbb{E}[u_c(\epsilon_o) R^i(\epsilon_o) | \epsilon_-], \\ \tau_o(\epsilon_o) &= 1 - \frac{u_x(\epsilon_o)}{u_c(\epsilon_o) f_L(\epsilon_o)}, \\ R_o b_o - \omega_o(\epsilon_o) + g_o(\epsilon_o) &\leq \tau_o(\epsilon_o) (1 - x_o(\epsilon_o)) f_L(\epsilon_o) + b_+(\epsilon_o), \\ c_o(\epsilon_o) + g_o(\epsilon_o) + k_+(\epsilon_o) &= (1 - \delta) k_o + f(\epsilon_o), \\ \underline{M}(\cdot) &\leq u_c(\epsilon_o) b_+(\epsilon_o) \leq \overline{M}(\cdot),\end{aligned}$$

where

$$\omega_o(\epsilon_o) \equiv \sum_i e_o^i (R^i(\epsilon_o) - R_o) + k_o^g (1 + f_K(\epsilon_o) - \delta - R_o)$$

denotes the return on the government's portfolio ($\{e_o^i\}_i, k_o^g$). The first three constraints represent the household's Euler equations for risk-free government debt, capital, and

the other assets. The fourth constraint relates the labor income tax rate to the household's marginal rate of substitution. The remaining constraints represent the (government) budget constraint, the resource constraint, and the debt limits.

Using the household's first-order conditions to substitute out τ_o and R_o and letting $\tilde{b}_o \equiv b_o \theta_-$, we can express the budget constraint as

$$\left(\frac{\tilde{b}_o}{\beta \mathbb{E}[u_c(\epsilon_o) | \epsilon_-]} - \tilde{\omega}_o(\epsilon_o) + g_o(\epsilon_o) \right) u_c(\epsilon_o) \leq (u_c(\epsilon_o) f_L(\epsilon_o) - u_x(\epsilon_o))(1 - x_o(\epsilon_o)) + \tilde{b}_+(\epsilon_o),$$

where $\tilde{\omega}_o(\epsilon_o)$ differs from $\omega_o(\epsilon_o)$ in that R_o is replaced by $\theta_- / (\beta \mathbb{E}[u_c(\epsilon_o) | \epsilon_-])$. The constraint set of the government is characterized by this modified budget constraint as well as the Euler equations, the resource constraint, and the debt limits. The Bellman equation reads

$$\begin{aligned} V(k_o, \tilde{b}_o, \theta_-, \epsilon_-) &= \max \mathbb{E}[u(\epsilon_o) + \beta V(k_+(\epsilon_o), \tilde{b}_+(\epsilon_o), u_c(\epsilon_o), \epsilon_o) | \epsilon_-] \\ \text{s.t.} & \quad \text{constraint set,} \end{aligned}$$

and the choice variables are $k_o^g, \{e_o^i\}_i, \{c_o(\epsilon_o), x_o(\epsilon_o), k_+(\epsilon_o), \tilde{b}_+(\epsilon_o)\}_{\epsilon_o}$. Note that in accordance with our definition of the state, the value function represents the unconditional value, "before" the realization of ϵ_o .

Let $\nu_o(\epsilon_o) \cdot \text{prob}(\epsilon_o | \epsilon_-)$ denote the multiplier associated with the budget constraint (the shadow value of public funds) when ϵ_o is realized. The government's first-order conditions with respect to $\tilde{b}_+(\epsilon_o)$ (assuming debt limits do not bind), e_o^i , and k_o^g , respectively, are given by

$$\begin{aligned} \nu_o(\epsilon_o) + \beta V_b(k_+(\epsilon_o), \tilde{b}_+(\epsilon_o), u_c(\epsilon_o), \epsilon_o) &= 0, \\ \mathbb{E}[\nu_o(\epsilon_o) u_c(\epsilon_o) (R^i(\epsilon_o) - R_o) | \epsilon_-] &= 0, \\ \mathbb{E}[\nu_o(\epsilon_o) u_c(\epsilon_o) (1 + f_K(\epsilon_o) - \delta - R_o) | \epsilon_-] &= 0, \end{aligned}$$

and the envelope condition implies

$$V_b(k_o, \tilde{b}_o, \theta_-, \epsilon_-) = - \sum_{\epsilon_o} \nu_o(\epsilon_o) \text{prob}(\epsilon_o | \epsilon_-) \frac{u_c(\epsilon_o)}{\beta \mathbb{E}[u_c(\epsilon_o) | \epsilon_-]} = - \frac{R_o}{\theta_-} \mathbb{E}[\nu_o(\epsilon_o) u_c(\epsilon_o) | \epsilon_-].$$

Combined, these equations yield the optimality conditions

$$\begin{aligned} \nu_-(\epsilon_-) \theta_- &= \beta \mathbb{E}[\nu_o(\epsilon_o) u_c(\epsilon_o) R_o | \epsilon_-], \\ \nu_-(\epsilon_-) \theta_- &= \beta \mathbb{E}[\nu_o(\epsilon_o) u_c(\epsilon_o) R^i(\epsilon_o) | \epsilon_-], \\ \nu_-(\epsilon_-) \theta_- &= \beta \mathbb{E}[\nu_o(\epsilon_o) u_c(\epsilon_o) (1 + f_K(\epsilon_o) - \delta) | \epsilon_-]. \end{aligned}$$

These conditions resemble the stochastic Euler equations characterizing a household's portfolio choice. They differ insofar as marginal utility is replaced by the product of marginal utility and the shadow value of public funds. Intuitively, the Ramsey policy equalizes the (average) valuation of public funds exactly as a household equalizes (average) marginal utility.

Note that the first condition coincides with the result derived earlier, namely that the change of the government budget multiplier, weighted by marginal utility, equals zero on average. This follows from

$$\mathbb{E} [\beta R_o v_o(\epsilon_o) u_c(\epsilon_o) - v_-(\epsilon_-) \theta_- | \epsilon_-] = \beta R_o \mathbb{E} [(v_o(\epsilon_o) - v_-(\epsilon_-)) u_c(\epsilon_o) | \epsilon_-] = 0.$$

The second and third optimality condition generalize this result. For any asset or liability in the government's portfolio, the Ramsey policy satisfies

$$\beta \mathbb{E} \left[v_o(\epsilon_o) u_c(\epsilon_o) \left(R^i(\epsilon_o) - R_o \right) | \epsilon_- \right] = 0.$$

That is, a more diversified portfolio results in a smoother multiplier and thus, better insurance for the government. If the portfolio were sufficiently diversified for the government (and the household) to face complete markets then the multiplier would be constant across histories.

The optimality conditions can be used to derive an asset pricing kernel for government projects. Letting $\mu_o(\epsilon_o) \cdot \text{prob}(\epsilon_o | \epsilon_-)$ denote the multiplier associated with the resource constraint when shock ϵ_o is realized, this kernel is given by

$$\beta \frac{v_o(\epsilon_o) u_c(\epsilon_o) + \mu_o(\epsilon_o)}{v_-(\epsilon_-) \theta_- + \mu_-(\epsilon_-)}.$$

12.2.3 Capital Income Taxation

A Neutrality Result Consider the model with capital of subsection 12.2.2 and assume that the government may impose state-contingent tax rates on the return on capital, in addition to labor income. Capital income tax rates, $\tau_o^k(\epsilon_o)$, only enter the household's Euler equation for capital and the government's budget constraint:

$$\begin{aligned} \theta_- &= \beta \mathbb{E} [u_c(\epsilon_o) (1 + (1 - \tau_o^k(\epsilon_o))(f_K(\epsilon_o) - \delta)) | \epsilon_-], \\ R_o b_o - \omega_o(\epsilon_o) + g_o(\epsilon_o) &\leq \tau_o(\epsilon_o) (1 - x_o(\epsilon_o)) f_L(\epsilon_o) + \tau_o^k(\epsilon_o) k_o (f_K(\epsilon_o) - \delta) + b_+(\epsilon_o). \end{aligned}$$

This implies a neutrality result: Optimal state-contingent capital income tax rates are indeterminate if the government faces complete markets, reflected in a portfolio with state-contingent returns, $\omega_o(\epsilon_o)$.

Intuitively, from the household's Euler equation, a redistribution of capital income taxes across states does not affect capital accumulation as long as the average tax wedge (suitably weighted) does not change. While such a redistribution alters the government's revenue in each of the affected states, this can be offset by adjusting the government portfolio as long as markets are complete. An equilibrium allocation thus can be implemented, for instance, either with state-contingent capital income taxes and bonds with risk-free coupons, or with risk-free capital income taxes and bonds with state-contingent coupons.

Zero Capital Income Taxation Consider a deterministic setting and suppose that the economy is inhabited by an infinitely lived, representative agent. The implementability constraint incorporates consumption and leisure sequences, $\{c_t, x_t\}_{t \geq 0}$, as well as the initial capital stock including interest, $k_0 R_0$, and tax rate, τ_0^k , (both predetermined). Write this constraint as

$$\iota_0(c_0, x_0, k_0 R_0, \tau_0^k) + \sum_{t=0}^{\infty} \beta^t \iota(c_t, x_t) = 0$$

for some functions ι and ι_0 . The Lagrangian associated with the Ramsey program reads

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \{v(c_t, x_t, v) - \mu_t(c_t + g_t + k_{t+1} - (1 - \delta)k_t - f(k_t, 1 - x_t))\} + v \iota_0(c_0, x_0, k_0 R_0, \tau_0^k),$$

where we define $v(c_t, x_t, v) \equiv u(c_t, x_t) + v \iota(c_t, x_t)$; v and $\beta^t \mu_t$ denote the multipliers associated with the implementability and resource constraints, respectively.

The first-order conditions with respect to $c_t, t \geq 1$, and k_{t+1} , respectively, are given by

$$\begin{aligned} v_c(c_t, x_t, v) &= \mu_t, \quad t \geq 1, \\ \mu_t &= \beta \mu_{t+1} (1 - \delta + f_K(k_{t+1}, 1 - x_{t+1})). \end{aligned}$$

Combining these conditions yields the key equation of interest which we report together with the household's Euler equation:

$$\begin{aligned} v_c(c_t, x_t, v) &= \beta (1 - \delta + f_K(k_{t+1}, 1 - x_{t+1})) v_c(c_{t+1}, x_{t+1}, v), \quad t \geq 1, \\ u_c(c_t, x_t) &= \beta (1 + (1 - \tau_{t+1}^k) (f_K(k_{t+1}, 1 - x_{t+1}) - \delta)) u_c(c_{t+1}, x_{t+1}). \end{aligned}$$

The two equations imply that under the Ramsey policy capital income is not taxed for $t \geq 2$ whenever $v_c(c_t, x_t, v)$ and $u_c(c_t, x_t)$ grow at the same rate. Two alternative assumptions guarantee such equal growth and thus, optimality of *zero capital income taxation*. The first assumption relates to preferences. If preferences are separable between consumption and leisure, and homothetic, then $v_c(c_t, x_t, v)$ is proportional to $u_c(c_t, x_t)$. In this case, the zero capital taxation result is an instance of the uniform commodity taxation result discussed in subsection 12.1.1. Recall from subsection 11.3.2 that capital income taxation is equivalent to time varying taxation of consumption. When consumption taxes are normalized to zero and the structure of preferences calls for uniform taxation of consumption, capital income therefore must not be taxed. Note that standard CIES preferences satisfy the separability and homotheticity condition.

The second, alternative assumption is the steady-state assumption. If the Ramsey allocation converges to a steady state then both $v_c(c_t, x_t, v)$ and $u_c(c_t, x_t)$ are constant over time and the optimality of zero capital income taxation follows. In fact, it also follows in richer environments as long as in steady state, the derivative of the implementability constraint(s) with respect to capital equal(s) zero and the multiplier(s) of the constraint(s) are constant. (This holds true, for example, in the steady state

of an economy with heterogenous households whose capital income—but not wage income—is taxed at a uniform rate.) Note that a strictly positive steady-state capital income tax rate would give rise to an ever increasing tax wedge between early and late consumption.

An upper bound on the tax rate, $\tau_{t+1}^k \leq \bar{\tau}^k$, which can be expressed as

$$1 - \left(\frac{u_c(c_t, x_t)}{\beta u_c(c_{t+1}, x_{t+1})} - 1 \right) / (f_K(k_{t+1}, 1 - x_{t+1}) - \delta) \leq \bar{\tau}^k,$$

introduces additional terms in the government's optimality conditions. As long as this constraint binds (forcing taxes to be spread over a longer horizon) the first key equation above contains additional terms, and this undermines the $\tau^k = 0$ implication. The constraint may bind forever.

12.2.4 Heterogeneous Households

When households are homogeneous the assumption that the government levies distorting taxes makes little sense: If everybody is the same, non-distorting lump-sum taxes clearly are preferable, and feasible. With heterogenous agents, in contrast, government policy is motivated by distributive concerns in addition to efficiency considerations. These concerns give rise to a trade-off between equity and efficiency and can rationalize taxation of an endogenous tax base.

Consider a variant of the economy analyzed in subsection 12.2.1 with two rather than one group of households, groups a and b with population shares η^a and $\eta^b = 1 - \eta^a$ respectively. Both groups have the same preferences but their labor productivities may differ, $w_t^a \neq w_t^b$. For simplicity, we abstract from risk. The resource constraint is given by

$$\eta^a c_t^a + \eta^b c_t^b + g_t = \eta^a w_t^a (1 - x_t^a) + \eta^b w_t^b (1 - x_t^b).$$

In each period, the government has two tax instruments at its disposal: A proportional labor income tax levied at rate τ_t , and a lump-sum tax, θ_t . The government's objective function is given by $\omega \eta^a U^a + \eta^b U^b$ where U^i denotes welfare of a member of group i and ω denotes some weight. The intertemporal budget constraint of a household in group $i = a, b$ reads

$$\sum_{t=0}^{\infty} q_t [c_t^i - (1 - \tau_t) w_t^i (1 - x_t^i) + \theta_t] = 0,$$

where q_t denotes the price at date $t = 0$ of a unit of the good at date t . The household first-order conditions

$$\begin{aligned} u_c(c_0^i, x_0^i) q_t &= \beta^t u_c(c_t^i, x_t^i), \\ u_c(c_t^i, x_t^i) w_t^i (1 - \tau_t) &= u_x(c_t^i, x_t^i) \end{aligned}$$

and the budget constraints yield the implementability constraints

$$\begin{aligned} \sum_{t=0}^{\infty} \beta^t [u_c(c_t^i, x_t^i)(c_t^i + \theta_t) - u_x(c_t^i, x_t^i)(1 - x_t^i)] &= 0, \\ u_c(c_t^a, x_t^a) / u_c(c_0^a, x_0^a) &= u_c(c_t^b, x_t^b) / u_c(c_0^b, x_0^b), \\ u_c(c_t^a, x_t^a) w_t^a / u_x(c_t^a, x_t^a) &= u_c(c_t^b, x_t^b) w_t^b / u_x(c_t^b, x_t^b). \end{aligned}$$

Let v^a and v^b denote the multipliers associated with the first two (intertemporal) implementability constraints. The optimal choice of lump-sum tax at date t satisfies

$$v^a u_c(c_t^a, x_t^a) + v^b u_c(c_t^b, x_t^b) = 0,$$

that is, the Ramsey policy sets the average multiplier equal to zero. To understand this result suppose first that group b did not exist such that only the first implementability constraint were present. The optimality condition for θ_t then would collapse to $v^a = 0$, indicating that the competitive equilibrium constraint is not binding for the Ramsey government. Intuitively, the Ramsey policy could implement the first best because the government could costlessly transfer resources from the private to the public sector.

With heterogeneous households, the lump-sum tax still allows the government to extract resources without distorting household choices. But since the lump-sum tax cannot be differentiated across groups the government generally does not reach first best (with respect to its objective function). Ideally, the government would costlessly transfer resources not only from the private to the public sector but also from the less to the more favored group. Since the latter is not possible, the choice of θ_t “at least” equalizes the average value of the multiplier with zero. If, by chance, the optimal lump-sum tax happens to implement the desired wealth distribution, then v^a and v^b individually equal zero as well. Otherwise, the Ramsey policy also employs labor income taxes, at the cost of generating tax distortions.

From the third implementability constraint, marginal utility grows at identical rates across groups. This implies that all but one lump-sum tax are redundant instruments (their first-order conditions are multiples of each other). With multiple lump-sum taxes, a Ricardian equivalence result applies: A change of timing of lump-sum taxes accompanied by suitable debt operations does not alter the equilibrium allocation.

To see how the timing of labor income taxes can affect the wealth distribution let $u(c, x) \equiv \ln(c) + \gamma \ln(x)$ and disregard lump-sum taxes. The implementability constraints then read

$$\begin{aligned} \sum_{t=0}^{\infty} \beta^t [1 - \gamma(1 - x_t^i) / x_t^i] &= 0, \\ \frac{c_0^a}{c_0^b} &= \frac{c_t^a}{c_t^b} = \frac{w_t^a x_t^a}{w_t^b x_t^b} \end{aligned}$$

or

$$\sum_{t=0}^{\infty} \beta^t \left[1 - \gamma \left(\frac{1}{x_t^a} - 1 \right) \right] = 0,$$

$$\sum_{t=0}^{\infty} \beta^t \left[1 - \gamma \left(\frac{c_0^a}{c_0^b} \frac{w_t^b}{w_t^a} \frac{1}{x_t^a} - 1 \right) \right] = 0.$$

From the first equation, raising x_0^a requires lowering x_t^a at some other date. From the second equation, this translates into a change of (time invariant) relative consumption, c_0^a/c_0^b , if relative productivity varies over time, $w_0^a/w_0^b \neq w_t^a/w_t^b$. Specifically, an increase in x_0^a (corresponding to a tax hike at date $t = 0$) and corresponding decrease of x_t^a raises c_0^a/c_0^b if $w_0^a/w_0^b \leq w_t^a/w_t^b$. Wealth is redistributed for two reasons. First, collecting taxes in periods where one group is relatively more productive shifts the tax burden to that group. Second, tax induced changes in interest rates affect debtors and creditors asymmetrically.

12.3 Social Insurance and Saving Taxation

When households are exposed to idiosyncratic risk the government can provide insurance by redistributing among agents ex post. When outcomes depend on households' efforts, however, then insurance undermines incentives, giving rise to a trade-off between insurance and incentives.

To analyze the consequences of this trade-off for the taxation of saving, consider a two-period economy with a continuum of measure one of ex ante identical households, indexed by i . Household i receives an endowment w_0 in the first period which can be consumed or saved at the gross interest rate R_1 . In the second period, the household works and consumes. Labor productivity, w_1^i , is random and i.i.d. across the population. Household i 's preferences are given by

$$U^i \equiv u(c_0^i) + \beta \mathbb{E}_0[u(c_1^i(\epsilon^1)) + v(x_1^i(\epsilon^1))],$$

where u and v are strictly increasing and concave.

Labor productivity and labor supply are private information. An insurance scheme in the second period thus can only be conditioned on (c_0^i, c_1^i, y_1^i) where $y_1^i \equiv w_1^i(1 - x_1^i)$ denotes labor income. The government maximizes the social welfare function $\int_i U^i di$ subject to the constraint

$$\left(w_0 - \int_i c_0^i di \right) R_1 + \int_i w_1^i (1 - x_1^i) di = \int_i c_1^i di$$

as well as incentive compatibility constraints which map (c_0^i, w_1^i) and the insurance scheme into the household's optimal labor supply.

Rather than studying a particular insurance scheme we adopt a general approach. By the *revelation principle* without loss of generality we may restrict attention to direct

mechanisms that induce truth telling. A mechanism maps household i 's self-reported productivity level, ρ_1^i , together with the observed c_0^i into (c_1^i, y_1^i) . It induces truth telling if households find it optimal to choose $\rho_1^i = w_1^i$ for all possible w_1^i .

Let (c_0^*, c_1^*, x_1^*) denote a consumption and leisure profile under the optimal mechanism and consider a marginal change of allocation that has no effect on households' incentives. Such a change results from a reduction of c_0 by $\Delta/u'(c_0^*)$ (more saving) where $\Delta > 0$ is small, and an increase of $c_1^*(w_1^i)$ by $\Delta/(\beta u'(c_1^*(w_1^i)))$ for all realizations w_1^i . Note that the policy change does not affect utility along the equilibrium path, for any w_1^i .

If the initial allocation is optimal then the policy change must be resource neutral, otherwise the government could have done better. Optimality thus requires

$$\frac{\Delta}{u'(c_0^*)} R_1 = \mathbb{E}_0 \left[\frac{\Delta}{\beta u'(c_1^*(w_1^i))} \right],$$

that is, the gain in second period resources (represented on the left-hand side of the equation) equals the loss (represented on the right-hand side, using the fact that w_1^i is i.i.d.). Cancelling Δ we have established that the optimal allocation satisfies a *reciprocal Euler equation*.

Absent incentive problems the government could perfectly insure consumption in the second period; dropping the expectations operator and inverting the condition would then yield the standard Euler equation. If the insurance scheme needs to provide incentives, in contrast, then consumption in the second period must be random (dependent on observed labor income and thus, unobserved productivity) and Jensen's inequality implies

$$u'(c_0^*) < \beta R_1 \mathbb{E}_0 \left[u'(c_1^*(w_1^i)) \right].$$

Since households can save at the risk-free interest rate R_1 this implies that the optimal mechanism must impose a tax on the return on saving.

Intuitively, with moral hazard, saving imposes a social cost, in addition to the cost and benefit internalized by a household. Specifically, a larger asset position reduces the covariance between labor supply and consumption in the second period and thus, undermines incentives. The optimal policy addresses this adverse effect by discouraging saving.

12.4 Optimal Monetary Policy

12.5 Bibliographic Notes

Ramsey (1927) introduces the primal approach. Atkinson and Stiglitz (1972) establish the results discussed in subsection 12.1.1. Diamond and Mirrlees (1971a; 1971b) introduce the dual approach and prove the production efficiency result. The many-person

Ramsey rule is due to Diamond (1975). Atkinson and Stiglitz (1976) show that commodity taxes are superfluous, with a linear labor income tax if specific conditions are satisfied; or with a nonlinear labor income tax as discussed by Mirrlees (1971) if preferences are weakly separable between consumption and leisure. The model presented in subsection 12.2.1 is due to Lucas and Stokey (1983). Angeletos (2002) shows that markets are complete even with non-contingent coupons as long as the maturity structure is sufficiently rich (see also Gale, 1990). Aiyagari, Marcet, Sargent and Seppälä (2002), Werning (2003), and Farhi (2010) analyze the model presented in subsection 12.2.2. The special case with linear utility provides micro foundations for Barro's (1979) and Bohn's (1990) tax-smoothing results. The indeterminacy result for optimal capital income taxes is due to Zhu (1992) and Chari, Christiano and Kehoe (1994). Chamley (1986) and Chari et al. (1994) derive the optimality of zero capital income taxes for CIES preferences. Chamley (1986) and Judd (1985) derive steady-state results and Straub and Werning (2014) clarify the conditions under which these results apply; see also Chari, Nicolini and Teles (2016). Farhi (2010) derives optimal capital income taxes in the model discussed in subsection 12.2.2. Subsection 12.2.4 follows Werning (2007) and Niepelt (2004*b*). Atkinson and Sandmo (1980) and King (1980) derive conditions for optimal capital income taxation in the OLG model. Diamond and Mirrlees (1978), Rogerson (1985), and Golosov, Kocherlakota and Tsyvinski (2003) establish the optimality of an intertemporal wedge in the presence of moral hazard.

Chapter 13

Equilibrium Policy

Chapter 14

Some Useful Models

Appendix A

Mathematical Tools

A.1 Constrained Optimization

Consider a maximization problem with one equality constraint and one inequality constraint:

$$\max_x f(x) \text{ s.t. } g(x) = 0, h(x) \leq 0.$$

Functions f, g, h are continuous and differentiable and $x \in \mathbb{R}^n$. Form the Lagrangian $\mathcal{L}(x, \lambda, \mu) \equiv f(x) - \lambda g(x) - \mu h(x)$.

Suppose that $x^* \in \mathbb{R}^n$ is a local maximizer of f on the constraint set. Suppose furthermore that the Jacobian matrix at x^* of the binding constraints has full rank. Then, there exist multipliers λ^* and μ^* such that $\partial \mathcal{L}(x^*, \lambda^*, \mu^*) / \partial x_i = 0$, $i = 1, \dots, n$; $\mu^* \geq 0$; $g(x^*) = 0$; $h(x^*) \leq 0$; and the complementary slackness condition $\mu^* h(x^*) = 0$ is satisfied.

Differentiating the Lagrangian with respect to the choice variables thus yields necessary conditions for a local maximum. The result extends to the case with multiple inequality and/or equality constraints.

A.2 Infinite-Horizon Dynamic Programming

The *sequence problem* is defined by

$$V^*(a_0) = \max_{\{a_{t+1}\}_{t \geq 0}} \sum_{t=0}^{\infty} \beta^t u(a_t R + w - a_{t+1}) \text{ s.t. } a_0 \text{ given, } a_{t+1} \in A(a_t). \quad (\text{SP})$$

Rather than imposing a no-Ponzi-game condition we require the choice variable to lie in the set described by the correspondence $A(a_t)$. A lower bound on this set implies that debt cannot be rolled over forever (if interest rates are strictly positive) and thus, rules out Ponzi games.

The *Bellman equation* associated with the sequence problem reads

$$V(a_t) = \max_{a_{t+1} \in A(a_t)} \{u(a_t R + w - a_{t+1}) + \beta V(a_{t+1})\} \text{ for all } a_t \in \mathcal{A}. \quad (\text{BE})$$

Since the problem is time-autonomous, the time indices of the state and the control variable in (BE) do not carry significance. \mathcal{A} denotes the state space.

A.2.1 Principle of Optimality

We assume that the set $A(a_t)$ is nonempty for all $a_t \in \mathcal{A}$. We also assume that for all sequences $\{a_{t+1}\}_{t \geq -1}$ that start with $a_0 \in \mathcal{A}$ and satisfy $a_{t+1} \in A(a_t)$, the infinite sequence in (SP) exists and is finite. Under these conditions, $V^*(a_t) = V(a_t)$ for all $a_t \in \mathcal{A}$. Moreover, a plan $\{a_{t+1}\}_{t \geq 0}$ conditional on $a_0 \in \mathcal{A}$ that attains $V^*(a_0)$ in (SP) also solves (BE), and the reverse statement holds as well; this is the *Principle of Optimality*.

The proofs of these results use the fact that if the infinite sum in (SP) exists and is finite, then it can be expressed as the sum of a contemporaneous payoff and a continuation payoff, similarly to the two terms on the right-hand side of (BE).

A.2.2 Uniqueness of V

If in addition, the set \mathcal{A} is bounded and complete; $A(a_t)$ is nonempty, bounded, and complete for all $a_t \in \mathcal{A}$; and A and u are continuous, then a unique continuous and bounded function V satisfying (BE) as well as an optimal plan $\{a_{t+1}\}_{t \geq 0}$ solving (SP) or (BE) exist.

The uniqueness result follows from mathematical theorems on *contractions*. Note that the right-hand side of (BE) constitutes an operator on the value function, $T(V)$ say: For any value function V on the right-hand side of (BE) the operator returns a value function on the left-hand side. The solution to (BE) then satisfies $V = T(V)$ and the function V constitutes a fixed point of the operator T .

Under the maintained assumptions, the maximization problem on the right-hand side of (BE) has a solution such that T exists and in fact, is continuous. The operator T therefore maps a set of continuous functions into the same set. Moreover, it satisfies Blackwell's sufficient conditions for a contraction.¹ But if an operator constitutes a contraction, then it has a unique fixed point. Moreover, repeated application of the operator generates a sequence of functions that converges to the fixed point. For an arbitrary continuous function V_0 , repeated application of the operator thus generates a sequence of functions, $V_0, T(V_0), T(T(V_0)), T(T(T(V_0))), \dots$, that converges to the fixed point V .

A.2.3 Properties of V

If in addition, u is concave and A convex, then the value function V in (BE) is strictly concave and the optimal plan $\{a_{t+1}\}_{t \geq 0}$ solving (SP) or (BE) is unique.

¹A metric space (\mathcal{M}, d) is a set \mathcal{M} whose elements can be added, multiplied by scalars, and for pairs of which a norm or distance d is defined. An operator T that maps a metric space into itself is a contraction if there exists a $\rho \in [0, 1)$ such that $d(T(m), T(n)) \leq \rho d(m, n)$ for all $m, n \in \mathcal{M}$.

If in addition, u is strictly increasing in the state and A is monotone, then the value function V in (BE) is strictly increasing.

If in addition, u is continuously differentiable on the interior of its domain, then the value function V in (BE) is differentiable.

A.3 Systems of Linear Difference Equations

Consider a system of two difference equations,

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} \quad \text{or} \quad z_{t+1} = Mz_t.$$

If M is diagonal (i.e., $b = c = 0$) then the two equations are uncoupled: we can solve them independently of each other, yielding $x_t = a^t x_0$ and $y_t = d^t y_0$ for arbitrary x_0, y_0 . If M is not diagonal, we can use eigenvalues and -vectors to transform the system and render the equations uncoupled.

Suppose that M has two distinct and real eigenvalues, ρ_1 and ρ_2 , with associated eigenvectors, v_1 and v_2 , satisfying

$$M[v_1 \ v_2] = [v_1 \ v_2] \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix} \quad \text{or} \quad MV = VP.$$

Pre-multiplying the original system $z_{t+1} = Mz_t$ by V^{-1} then yields a transformed, uncoupled system in the vector $Z \equiv V^{-1}z$ with diagonal matrix entries equal to the eigenvalues of M :

$$Z_{t+1} = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix} Z_t \quad \text{and therefore} \quad Z_t = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix}^t Z_0.$$

Using $z_t = VZ_t$, the latter equation can be transformed back to yield

$$z_t = V \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix}^t Z_0 = V \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix}^t V^{-1}z_0.$$

Note that M^t is given by VP^tV^{-1} . Letting $\varphi_0 \equiv V^{-1}z_0$ we conclude that

$$z_t = V \begin{bmatrix} \rho_1^t & 0 \\ 0 & \rho_2^t \end{bmatrix} \varphi_0 \quad \text{or} \quad z_t = \varphi_{0[1]} \rho_1^t v_1 + \varphi_{0[2]} \rho_2^t v_2.$$

This first-order difference equation system in z has a family of solutions with two degrees of freedom, corresponding to the two elements of z_0 or φ_0 . For a definite solution, we need two restrictions. An initial condition for an element of z_0 constitutes such a restriction. If an eigenvalue is unstable then the requirement that system dynamics be stable also implies a restriction; for example, $\rho_1 > 1$ and system stability imply $\varphi_{0[1]} = 0$.

If the eigenvalues of M are not distinct or if they are complex then the solution strategy must be adapted. The extension to the case with more than two variables is immediate.

A.4 Bibliographic Notes

Simon and Blume (1994, 18, 19), Mas-Colell et al. (1995, M.K), and Acemoglu (2009, A.11) review Lagrangian methods. Stokey and Lucas (1989, 3, 4) and Acemoglu (2009, 6) contain detailed proofs and discussions on dynamic programming. Acemoglu (2009, Example 6.5) covers the saving problem and discusses an approach to guarantee compactness of \mathcal{A} in that program. Simon and Blume (1994, 23) cover linear difference equations, including the case of repeated or complex eigenvalues.

Beyond the material covered in the chapter, Stokey and Lucas (1989, 7–9) and Acemoglu (2009, 16) cover dynamic programming under risk.

Appendix B

Technical Discussions

B.1 Transversality Condition in Infinite-Horizon Saving Problem

The household's program is given by

$$\max_{\{a_{t+1}\}_{t \geq 0}} \sum_{t=0}^{\infty} \beta^t u(a_t R_t + w_t - a_{t+1}) \quad \text{s.t. } a_0 \text{ given, } \lim_{T \rightarrow \infty} q_T a_{T+1} \geq 0.$$

Let $\hat{a} \equiv \{\hat{a}_{t+1}\}_{t \geq 0}$ denote a plan that satisfies the Euler equation at all times as well as $\lim_{T \rightarrow \infty} q_T \hat{a}_{T+1} = 0$. Let $\bar{a} \equiv \{\bar{a}_{t+1}\}_{t \geq 0}$ denote an alternative plan that satisfies $\lim_{T \rightarrow \infty} q_T \bar{a}_{T+1} \geq 0$. We want to show that the former dominates the latter.

For brevity, let $\hat{u}_t \equiv u(\hat{a}_t R_t + w_t - \hat{a}_{t+1})$ and $\hat{u}'_t \equiv u'(\hat{a}_t R_t + w_t - \hat{a}_{t+1})$ and similarly for \bar{u}_t and \bar{u}'_t . Strict concavity of u and positive marginal utility imply $\hat{u}_t + R_t \hat{u}'_t(\bar{a}_t - \hat{a}_t) - \hat{u}'_t(\bar{a}_{t+1} - \hat{a}_{t+1}) > \bar{u}_t$ if $\hat{a}_t \neq \bar{a}_t$ or $\hat{a}_{t+1} \neq \bar{a}_{t+1}$. If $\hat{a}_{t+1} \neq \bar{a}_{t+1}$ for some $t \in \{0, \dots, T\}$, we thus have

$$\sum_{t=0}^T \beta^t (\bar{u}_t - \hat{u}_t) < \sum_{t=0}^T \beta^t \{R_t \hat{u}'_t(\bar{a}_t - \hat{a}_t) - \hat{u}'_t(\bar{a}_{t+1} - \hat{a}_{t+1})\} = \beta^T \hat{u}'_T(\hat{a}_{T+1} - \bar{a}_{T+1}),$$

where we use $\hat{a}_0 = \bar{a}_0$ and $\hat{u}'_t = \beta R_{t+1} \hat{u}'_{t+1}$. From the Euler equation, $\beta^T \hat{u}'_T = \hat{u}'_0 q_T$. If \hat{a} is not identical to \bar{a} , it follows that

$$\lim_{T \rightarrow \infty} \sum_{t=0}^T \beta^t (\bar{u}_t - \hat{u}_t) < \lim_{T \rightarrow \infty} \hat{u}'_0 q_T (\hat{a}_{T+1} - \bar{a}_{T+1}) = \lim_{T \rightarrow \infty} -\hat{u}'_0 q_T \bar{a}_{T+1} \leq 0$$

such that $\sum_{t=0}^{\infty} \beta^t \hat{u}_t > \sum_{t=0}^{\infty} \beta^t \bar{u}_t$. Satisfying $\lim_{T \rightarrow \infty} q_T \hat{a}_{T+1} = 0$ therefore is optimal.

B.2 Representative Household

B.3 Transversality Condition in Infinite-Horizon Planner Problem

Let $g(k_t) \equiv k_t(1 - \delta) + f(k_t, 1)$ with f denoting the neoclassical production function. Note that the function g is strictly concave. The program is given by

$$\max_{\{k_{t+1}\}_{t \geq 0}} \sum_{t=0}^{\infty} \beta^t u(g(k_t) - k_{t+1}) \quad \text{s.t. } k_0 \text{ given, } k_{t+1} \geq 0.$$

Let $\hat{k} \equiv \{\hat{k}_{t+1}\}_{t \geq 0}$ denote a plan that satisfies the Euler equation and complementary slackness condition, $\hat{u}'_t = \beta g'(\hat{k}_{t+1}) \hat{u}'_{t+1} + \hat{\mu}_t$ and $\hat{\mu}_t \hat{k}_{t+1} = 0$ respectively, at all times, as well as $\lim_{T \rightarrow \infty} \beta^T \hat{u}'_T \hat{k}_{T+1} = 0$. Let $\bar{k} \equiv \{\bar{k}_{t+1}\}_{t \geq 0}$ denote an alternative plan that satisfies $\lim_{T \rightarrow \infty} \beta^T \bar{u}'_T \bar{k}_{T+1} \geq 0$. Here, we let $\hat{u}_t \equiv u(g(\hat{k}_t) - \hat{k}_{t+1})$ and $\hat{u}'_t \equiv u'(g(\hat{k}_t) - \hat{k}_{t+1})$ and similarly for \bar{u}_t and \bar{u}'_t . We want to show that \hat{k} dominates \bar{k} .

Strict concavity of u and g and positive marginal utility imply $\hat{u}_t + \hat{u}'_t g'(\hat{k}_t)(\bar{k}_t - \hat{k}_t) - \hat{u}'_t(\bar{k}_{t+1} - \hat{k}_{t+1}) > \bar{u}_t$ if $\hat{k}_t \neq \bar{k}_t$ or $\hat{k}_{t+1} \neq \bar{k}_{t+1}$. If $\hat{k}_{t+1} \neq \bar{k}_{t+1}$ for some $t \in \{0, \dots, T\}$, we thus have

$$\begin{aligned} \sum_{t=0}^T \beta^t (\bar{u}_t - \hat{u}_t) &< \sum_{t=0}^T \beta^t \{ \hat{u}'_t g'(\hat{k}_t)(\bar{k}_t - \hat{k}_t) - \hat{u}'_t(\bar{k}_{t+1} - \hat{k}_{t+1}) \} \\ &= \beta^T \hat{u}'_T (\hat{k}_{T+1} - \bar{k}_{T+1}) - \sum_{t=0}^{T-1} \beta^t \hat{\mu}_t \bar{k}_{t+1}, \end{aligned}$$

where we use $\hat{k}_0 = \bar{k}_0$, the Euler equation, and the complementary slackness condition. If \hat{k} is not identical to \bar{k} , it follows that

$$\begin{aligned} \lim_{T \rightarrow \infty} \sum_{t=0}^T \beta^t (\bar{u}_t - \hat{u}_t) &< \lim_{T \rightarrow \infty} \left\{ \beta^T \hat{u}'_T (\hat{k}_{T+1} - \bar{k}_{T+1}) - \sum_{t=0}^{T-1} \beta^t \hat{\mu}_t \bar{k}_{t+1} \right\} \\ &= - \lim_{T \rightarrow \infty} \left\{ \beta^T \hat{u}'_T \bar{k}_{T+1} + \sum_{t=0}^{T-1} \beta^t \hat{\mu}_t \bar{k}_{t+1} \right\} \leq 0, \end{aligned}$$

since marginal utility, the capital stock, and the multiplier all are weakly positive. We conclude that $\sum_{t=0}^{\infty} \beta^t \hat{u}_t > \sum_{t=0}^{\infty} \beta^t \bar{u}_t$. Satisfying $\lim_{T \rightarrow \infty} \beta^T \hat{u}'_T \hat{k}_{T+1} = 0$ therefore is optimal.

B.4 Non-Expected Utility

B.5 Linear Rational Expectations Models

Consider a linear rational expectations model represented by the system of difference equations

$$\begin{bmatrix} x_{t+1} \\ \mathbb{E}_t[y_{t+1}] \end{bmatrix} = M \begin{bmatrix} x_t \\ y_t \end{bmatrix} + Ns_t \quad (\text{B.1})$$

or equivalently,

$$\begin{bmatrix} x_{t+1} \\ (n_x \times 1) \\ y_{t+1} \\ (n_y \times 1) \end{bmatrix} = \begin{matrix} M \\ (n \times n) \end{matrix} \begin{bmatrix} x_t \\ (n_x \times 1) \\ y_t \\ (n_y \times 1) \end{bmatrix} + \begin{matrix} N \\ (n \times n_s)(n_s \times 1) \end{matrix} s_t + \begin{bmatrix} 0 \\ (n_x \times 1) \\ f_{t+1} \\ (n_y \times 1) \end{bmatrix}.$$

The model has n_x predetermined variables (including for example the capital stock), denoted by x ; n_y non-predetermined variables (e.g., consumption), denoted by y ; and n_s exogenous variables (e.g., productivity), denoted by s . The number of endogenous variables equals $n = n_x + n_y$, and f denotes a vector of forecast errors.

Using the notation of appendix A.3, the matrix M can be represented as the product of matrices containing its eigenvectors and eigenvalues,

$$M = VPV^{-1} \equiv \begin{matrix} [v_1 \ v_2 \ \dots \ v_n] \\ (n \times n) \end{matrix} \begin{bmatrix} \rho_1 & & 0 \\ & \ddots & \\ 0 & & \rho_n \end{bmatrix} \begin{matrix} V^{-1} \\ (n \times n) \end{matrix} \equiv V \begin{bmatrix} P_{<<} & 0 \\ (n_{<} \times n_{<}) & (n_{<} \times n_{>}) \\ 0 & P_{>>} \\ (n_{>} \times n_{<}) & (n_{>} \times n_{>}) \end{bmatrix} V^{-1},$$

where $n_{<}$ and $n_{>}$ denote the number of eigenvalues whose absolute value is weakly smaller than unity or exceeds unity, respectively. We assume that the eigenvalues are distinct and real and ordered in ascending absolute value, $|\rho_1| \leq |\rho_2| \leq \dots \leq |\rho_n|$. Let

$$Z_t \equiv \begin{bmatrix} Z_{< t} \\ (n_{<} \times 1) \\ Z_{> t} \\ (n_{>} \times 1) \end{bmatrix} \equiv V^{-1} \begin{bmatrix} x_t \\ y_t \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} N_{<} \\ (n_{<} \times n_s) \\ N_{>} \\ (n_{>} \times n_s) \end{bmatrix} \equiv V^{-1}N. \quad (\text{B.2})$$

Premultiplying equation (B.1) by V^{-1} and iterating the equation forward yields

$$\mathbb{E}_t[Z_{t+i}] = P^i Z_t + \sum_{j=0}^{i-1} P^{i-1-j} V^{-1} N \mathbb{E}_t[s_{t+j}].$$

Stability ($\lim_{i \rightarrow \infty} \mathbb{E}_t Z_{t+i} = 0$) thus requires

$$Z_{> t} = - \sum_{j=0}^{\infty} (P_{>>})^{-1-j} N_{>} \mathbb{E}_t[s_{t+j}].$$

That is, the requirement that system dynamics be stable imposes $n_{>}$ restrictions on Z_t . The initial conditions for the predetermined variables x_t impose n_x additional restrictions. We may distinguish three cases:

B.5.1 No Solution, $n_y < n_>$

If $n_y < n_>$ (and thus, $n_x > n_<$) then the stability requirement imposes more restrictions than there are non-predetermined variables that could adjust to satisfy them. In general, the model has no solution in this case.

B.5.2 Determinacy, $n_y = n_>$

If $n_y = n_>$ then the stability requirement imposes as many restrictions as there are non-predetermined variables. All endogenous variables thus are uniquely pinned down (as long as the relevant submatrix of V^{-1} is invertible such that a given $Z_{>t}$ implies a unique y_t in (B.2)) and model dynamics satisfy

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = V \begin{bmatrix} P_{<<} & 0 \\ 0 & 0 \end{bmatrix} V^{-1} \begin{bmatrix} x_t \\ y_t \end{bmatrix} + V \begin{bmatrix} N_{<} \\ 0 \end{bmatrix} s_t + V \begin{bmatrix} 0 \\ (n_{<} \times n_{>}) \\ I \\ (n_{>} \times n_{>}) \end{bmatrix} Z_{>t+1}.$$

The realized values of the non-predetermined variables typically differ from their expected values, $y_{t+1} \neq \mathbb{E}_t[y_{t+1}]$. The difference, f_{t+1} , solely reflects the fact that the exogenous variables differ from their expected values, $s_{t+1} \neq \mathbb{E}_t[s_{t+1}]$, and thus, $Z_{>t+1} \neq \mathbb{E}_t[Z_{>t+1}]$.

B.5.3 Indeterminacy, $n_y > n_>$

If $n_y > n_>$ then the stability requirement does not uniquely pin down the non-predetermined variables; there are $n_y - n_>$ degrees of freedom. Suppose, for example, that $n_x = n_s = 0$ such that the model reduces to

$$y_{t+1} = My_t + f_{t+1}$$

and $Z_{>t} = 0$. In this case, $n_{<}$ elements of y_t can freely be chosen in each period without triggering explosive dynamics. Although $s_t = 0$ in all periods the forecast errors may be non-zero. In particular, y_t may respond to—fundamentally irrelevant—“sunspot” shocks.

B.6 Bibliographic Notes

Sargent and Wallace (1973) discuss the forward solution and Blanchard and Kahn (1980) propose the solution strategy for linear rational expectations models, following Vaughan (1970). Miao (2014, 2) reviews alternative solution strategies.

Bibliography

- Acemoglu, D. (2009), *Introduction to Modern Economic Growth*, Princeton University Press, Princeton.
- Aiyagari, S. R. (1994), 'Uninsured idiosyncratic risk and aggregate saving', *Quarterly Journal of Economics* **109**(3), 659–684.
- Aiyagari, S. R. and Gertler, M. (1985), 'The backing of government bonds and Monetarism', *Journal of Monetary Economics* **16**(1), 19–44.
- Aiyagari, S. R., Marcet, A., Sargent, T. J. and Seppälä, J. (2002), 'Optimal taxation without state-contingent debt', *Journal of Political Economy* **110**(6), 1220–1254.
- Allais, M. (1947), *Economie et Interet*, Imprimerie Nationale, Paris.
- Alvarez, F. and Jermann, U. J. (2004), 'Using asset prices to measure the cost of business cycles', *Journal of Political Economy* **112**(6), 1223–1256.
- Angeletos, G.-M. (2002), 'Fiscal policy with noncontingent debt and the optimal maturity structure', *Quarterly Journal of Economics* **117**(3), 1105–1131.
- Arrow, K. J. and Debreu, G. (1954), 'Existence of an equilibrium for a competitive economy', *Econometrica* **22**(3), 265–290.
- Atkinson, A. B. and Sandmo, A. (1980), 'Welfare implications of the taxation of savings', *Economic Journal* **90**(359), 529–549.
- Atkinson, A. B. and Stiglitz, J. E. (1972), 'The structure of indirect taxation and economic efficiency', *Journal of Public Economics* **1**(1), 97–119.
- Atkinson, A. B. and Stiglitz, J. E. (1976), 'The design of tax structure: Direct versus indirect taxation', *Journal of Public Economics* **6**(1–2), 55–75.
- Auerbach, A. J., Gokhale, J. and Kotlikoff, L. J. (1994), 'Generational accounting: A meaningful way to evaluate fiscal policy', *Journal of Economic Perspectives* **8**(1), 73–94.
- Backus, D. K. and Smith, G. W. (1993), 'Consumption and real exchange rates in dynamic economies with non-traded goods', *Journal of International Economics* **35**(3–4), 297–316.

- Balassa, B. (1964), 'The purchasing-power parity doctrine: A reappraisal', *Journal of Political Economy* **72**(6), 584–596.
- Ball, L. and Mankiw, N. G. (2007), 'Intergenerational risk sharing in the spirit of Arrow, Debreu, and Rawls, with applications to social security design', *Journal of Political Economy* **115**(4), 523–547.
- Barro, R. J. (1974), 'Are government bonds net wealth?', *Journal of Political Economy* **82**(6), 1095–1117.
- Barro, R. J. (1979), 'On the determination of the public debt', *Journal of Political Economy* **87**(5), 940–971.
- Barro, R. J. (1990), 'Government spending in a simple model of endogenous growth', *Journal of Political Economy* **98**(5), S103–S125.
- Barro, R. J. and Sala-i-Martin, X. (1995), *Economic Growth*, McGraw-Hill, New York.
- Bassetto, M. and Kocherlakota, N. (2004), 'On the irrelevance of government debt when taxes are distortionary', *Journal of Monetary Economics* **51**(2), 299–304.
- Baxter, M. and King, R. G. (1993), 'Fiscal policy in general equilibrium', *American Economic Review* **83**, 315–334.
- Becker, G. S. (1965), 'A theory of the allocation of time', *Economic Journal* **75**(299), 493–517.
- Benhabib, J. and Farmer, R. E. A. (1994), 'Indeterminacy and increasing returns', *Journal of Economic Theory* **63**(1), 19–41.
- Bewley, T. F. (1977), 'The permanent income hypothesis: A theoretical formulation', *Journal of Economic Theory* **16**(2), 252–292.
- Bewley, T. F. (1980), The optimum quantity of money, in J. H. Kareken and N. Wallace, eds, 'Models of Monetary Economies', Federal Reserve Bank of Minneapolis, Minneapolis, pp. 169–210.
- Bewley, T. F. (1986), Stationary monetary equilibrium with a continuum of independently fluctuating consumers, in W. Hildenbrand and A. Mas-Colell, eds, 'Contributions to Mathematical Economics in Honor of Gerard Debreu', North Holland, Amsterdam, pp. 79–102.
- Blanchard, O. J. (1985), 'Debt, deficits, and finite horizons', *Journal of Political Economy* **93**(2), 223–247.
- Blanchard, O. J. (2000), 'What do we know about macroeconomics that Fisher and Wicksell did not?', *Quarterly Journal of Economics* **115**(4), 1375–1410.

- Blanchard, O. J. and Kahn, C. M. (1980), 'The solution of linear difference models under rational expectations', *Econometrica* **48**(5), 1305–1311.
- Blanchard, O. J. and Weil, P. (1992), Dynamic efficiency, the riskless rate, and debt ponzi games under uncertainty, Working Paper 3992, NBER, Cambridge, Massachusetts.
- Bohn, H. (1990), 'Tax smoothing with financial instruments', *American Economic Review* **80**(5), 1217–1230.
- Breeden, D. T. (1979), 'An intertemporal asset pricing model with stochastic consumption and investment opportunities', *Journal of Financial Economics* **7**(3), 265–296.
- Breyer, F. (1989), 'On the intergenerational Pareto efficiency of pay-as-you-go financed pension systems', *Journal of Institutional and Theoretical Economics* **145**(4), 643–658.
- Brock, W. A. and Mirman, L. J. (1972), 'Optimal economic growth and uncertainty: The discounted case', *Journal of Economic Theory* **4**(3), 479–513.
- Buiter, W. H. (1981), 'Time preference and international lending and borrowing in an overlapping-generations model', *Journal of Political Economy* **89**(4), 769–797.
- Buiter, W. H. (2002), 'The fiscal theory of the price level: A critique', *Economic Journal* **112**(481), 459–480.
- Cagan, P. (1956), The monetary dynamics of hyperinflation, in M. Friedman, ed., 'Studies in the Quantity Theory of Money', University of Chicago Press, Chicago, pp. 25–117.
- Carroll, C. D. (1997), 'Buffer-stock saving and the life cycle/permanent income hypothesis', *Quarterly Journal of Economics* **112**(1), 1–55.
- Cass, D. (1965), 'Optimum growth in an aggregative model of capital accumulation', *Review of Economic Studies* **32**, 233–240.
- Cass, D. (1972), 'On capital overaccumulation in the aggregative, neoclassical model of economic growth: A complete characterization', *Journal of Economic Theory* **4**, 200–223.
- Chamberlain, G. and Wilson, C. A. (2000), 'Optimal intertemporal consumption under uncertainty', *Review of Economic Dynamics* **3**(3), 365–395.
- Chamley, C. (1986), 'Optimal taxation of capital income in general equilibrium with infinite lives', *Econometrica* **54**(3), 607–622.
- Chamley, C. and Polemarchakis, H. (1984), 'Assets, general equilibrium and the neutrality of money', *Review of Economic Studies* **51**(1), 129–138.

- Chari, V., Nicolini, J. P. and Teles, P. (2016), More on the optimal taxation of capital. Mimeo, Federal Reserve Bank of Minneapolis.
- Chari, V. V., Christiano, L. J. and Kehoe, P. J. (1994), 'Optimal fiscal policy in a business cycle model', *Journal of Political Economy* **102**(4), 617–652.
- Cobb, C. W. and Douglas, P. H. (1928), 'A theory of production', *American Economic Review, Papers and Proceedings* **18**(1), 139–165.
- Cochrane, J. H. (2001), *Asset Pricing*, Princeton University Press, Princeton.
- Cooley, T. F., ed. (1995), *Frontiers of Business Cycle Research*, Princeton University Press, Princeton.
- Deaton, A. (1991), 'Saving and liquidity constraints', *Econometrica* **59**(5), 1221–1248.
- Debreu, G. (1959), *Theory of Value*, John Wiley, New York.
- Diamond, P. A. (1965), 'National debt in a neoclassical growth model', *American Economic Review* **55**(5), 1126–1150.
- Diamond, P. A. (1975), 'A many-person Ramsey tax rule', *Journal of Public Economics* **4**(4), 335–342.
- Diamond, P. A. (1982), 'Aggregate demand management in search equilibrium', *Journal of Political Economy* **90**, 881–894.
- Diamond, P. A. and Mirrlees, J. A. (1971a), 'Optimal taxation and public production I: Production efficiency', *American Economic Review* **61**(1), 8–27.
- Diamond, P. A. and Mirrlees, J. A. (1971b), 'Optimal taxation and public production II: Tax rules', *American Economic Review* **61**(3), 261–278.
- Diamond, P. A. and Mirrlees, J. A. (1978), 'A model of social insurance with variable retirement', *Journal of Public Economics* **10**(3), 295–336.
- Dixit, A. (1989), 'Entry and exit decisions under uncertainty', *Journal of Political Economy* **97**(3), 620–638.
- Dornbusch, R., Fischer, S. and Samuelson, P. A. (1977), 'Comparative advantage, trade, and payments in a Ricardian model with a continuum of goods', *American Economic Review* **67**(5), 823–839.
- Farhi, E. (2010), 'Capital taxation and ownership when markets are incomplete', *Journal of Political Economy* **118**(5), 908–948.
- Fried, J. (1980), 'The intergenerational distribution of the gains from technical change and from international trade', *The Canadian Journal of Economics* **13**(1), 65–81.

- Friedman, M. (1957), *A Theory of the Consumption Function*, Princeton University Press, Princeton.
- Friedman, M. (1968), 'The role of monetary policy', *American Economic Review* **58**(1), 1–17.
- Gale, D. (1990), The efficient design of public debt, in R. Dornbusch and M. Draghi, eds, 'Public Debt Management: Theory and History', Cambridge University Press, Cambridge, England, chapter 2, pp. 14–47.
- Golosov, M., Kocherlakota, N. and Tsyvinski, A. (2003), 'Optimal indirect and capital taxation', *Review of Economic Studies* **70**, 569–587.
- Gonzalez-Eiras, M. and Niepelt, D. (2015), 'Politico-economic equivalence', *Review of Economic Dynamics* **18**(4), 843–862.
- Hall, R. E. (1978), 'Stochastic implications of the life cycle-permanent income hypothesis: Theory and evidence', *Journal of Political Economy* **86**(6), 971–987.
- Hall, R. E. (2005), 'Employment fluctuations with equilibrium wage stickiness', *American Economic Review* **95**(1), 50–65.
- Hansen, G. D. (1985), 'Indivisible labor and the business cycle', *Journal of Monetary Economics* **16**(3), 309–327.
- Harrod, R. F. (1933), *International Economics*, Cambridge University Press, Cambridge.
- Hayashi, F. (1982), 'Tobin's marginal q and average q: A neoclassical interpretation', *Econometrica* **50**(1), 213–224.
- Hosios, A. J. (1990), 'On the efficiency of matching and related models of search and unemployment', *Review of Economic Studies* **57**(2), 279–298.
- Huggett, M. (1993), 'The risk-free rate in heterogeneous-agent incomplete-insurance economies', *Journal of Economic Dynamics and Control* **17**(5–6), 953–969.
- Judd, K. L. (1985), 'Redistributive taxation in a simple perfect foresight model', *Journal of Public Economics* **28**, 59–83.
- Kaldor, N. (1961), Capital accumulation and economic growth, in F. A. Lutz and D. C. Hague, eds, 'The Theory of Capital', St. Martins Press, New York, pp. 177–222.
- Kimball, M. S. (1990), 'Precautionary saving in the small and in the large', *Econometrica* **58**(1), 53–73.
- King, M. A. (1980), Savings and taxation, in G. Hughes and G. Heal, eds, 'Public Policy and the Tax System', George Allen & Unwin, London, chapter 1, pp. 1–35.
- King, R. G., Plosser, C. I. and Rebelo, S. T. (1988), 'Production, growth, and business cycles I: The basic neoclassical model', *Journal of Monetary Economics* **21**, 195–232.

- King, R. G., Plosser, C. I. and Rebelo, S. T. (2002), 'Production, growth, and business cycles: Technical appendix', *Computational Economics* **20**(1), 87–116.
- Kocherlakota, N. and Phelan, C. (1999), 'Explaining the fiscal theory of the price level', *Federal Reserve Bank of Minneapolis Quarterly Review* **23**(4), 14–23.
- Koopmans, T. C. (1965), On the concept of optimal economic growth, in 'The Econometric Approach to Development Planning', North-Holland / Rand McNally, Amsterdam, chapter 4, pp. 225–300. Reissue of *Pontificiae Academiae Scientiarum Scripta Varia* 28.
- Krusell, P. and Smith, A. A. (1998), 'Income and wealth heterogeneity in the macroeconomy', *Journal of Political Economy* **106**(5), 867–896.
- Kydland, F. E. and Prescott, E. C. (1982), 'Time to build and aggregate fluctuations', *Econometrica* **50**(6), 1345–1370.
- Laibson, D. (1997), 'Golden eggs and hyperbolic discounting', *Quarterly Journal of Economics* **62**, 443–477.
- Leeper, E. M. (1991), 'Equilibria under 'active' and 'passive' monetary and fiscal policies', *Journal of Monetary Economics* **27**(1), 129–147.
- Leland, H. E. (1968), 'Saving and uncertainty: The precautionary demand for saving', *Quarterly Journal of Economics* **82**(3), 465–473.
- LeRoy, S. F. and Werner, J. (2014), *Principles of Financial Economics*, Cambridge University Press, Cambridge.
- Lintner, J. (1965), 'The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets', *Review of Economics and Statistics* **47**(1), 13–37.
- Long, J. B. and Plosser, C. I. (1983), 'Real business cycles', *Journal of Political Economy* **91**(11), 39–69.
- Lucas, R. E. (1976), 'Econometric policy evaluation: A critique', *Carnegie-Rochester Conference Series on Public Policy* pp. 19–46.
- Lucas, R. E. (1978), 'Asset prices in an exchange economy', *Econometrica* **46**(6), 1429–1445.
- Lucas, R. E. (1987), *Models of Business Cycles*, Basil Blackwell, New York.
- Lucas, R. E. and Prescott, E. C. (1971), 'Investment under uncertainty', *Econometrica* **39**(5), 659–681.
- Lucas, R. E. and Rapping, L. A. (1969), 'Real wages, employment, and inflation', *Journal of Political Economy* **77**(5), 721–754.

- Lucas, R. E. and Stokey, N. L. (1983), 'Optimal fiscal and monetary policy in an economy without capital', *Journal of Monetary Economics* **12**(1), 55–93.
- Magill, M. and Quinzii, M. (1996), *Theory of Incomplete Markets*, Vol. 1, MIT Press, Cambridge, Massachusetts.
- Mas-Colell, A., Whinston, M. D. and Green, J. R. (1995), *Microeconomic Theory*, Oxford University Press, New York.
- Mehra, R. and Prescott, E. C. (1985), 'The equity premium: A puzzle', *Journal of Monetary Economics* **15**(2), 145–161.
- Merz, M. (1995), 'Search in the labor market and the real business cycle', *Journal of Monetary Economics* **36**(2), 269–300.
- Miao, J. (2014), *Economic Dynamics in Discrete Time*, MIT Press, Cambridge, Massachusetts.
- Mirrlees, J. A. (1971), 'An exploration in the theory of optimum income taxation', *Review of Economic Studies* **38**(114), 175–208.
- Modigliani, F. and Brumberg, R. (1954), Utility analysis and the consumption function: An interpretation of cross-section data, in K. K. Kurihara, ed., 'Post Keynesian Economics', Rutgers University Press, New Brunswick.
- Modigliani, F. and Miller, M. H. (1958), 'The cost of capital, corporation finance and the theory of investment', *American Economic Review* **48**(3), 261–297.
- Mortensen, D. T. (1982), 'Property rights and efficiency in mating, racing, and related games', *American Economic Review* **72**(5), 968–979.
- Mossin, J. (1966), 'Equilibrium in a capital asset market', *Econometrica* **34**(4), 768–783.
- Muth, J. F. (1961), 'Rational expectations and the theory of price movements', *Econometrica* **29**(3), 315–335.
- Niepelt, D. (2004a), 'The fiscal myth of the price level', *Quarterly Journal of Economics* **119**(1), 277–300.
- Niepelt, D. (2004b), 'Tax smoothing versus tax shifting', *Review of Economic Dynamics* **7**(1), 27–51.
- Obstfeld, M. (1982), 'Aggregate spending and the terms of trade: Is there a Laursen-Metzler effect?', *Quarterly Journal of Economics* **97**(2), 251–270.
- Phelps, E. S., ed. (1970), *Microeconomic Foundations of Employment and Inflation Theory*, Norton, New York.
- Pissarides, C. A. (1985), 'Short-run equilibrium dynamics of unemployment, vacancies, and real wages', *American Economic Review* **75**(4), 676–690.

- Pissarides, C. A. (1990), *Equilibrium Unemployment Theory*, Basil Blackwell, Oxford.
- Pissarides, C. A. (2000), *Equilibrium Unemployment Theory*, MIT Press, Cambridge, Massachusetts.
- Radner, R. (1982), Equilibrium under uncertainty, in K. J. Arrow and M. D. Intriligator, eds, 'Handbook of Mathematical Economics', Vol. 2, North-Holland, chapter 20, pp. 923–1006.
- Ramsey, F. P. (1927), 'A contribution to the theory of taxation', *Economic Journal* 37(145), 47–61.
- Ramsey, F. P. (1928), 'A mathematical theory of saving', *Economic Journal* 38(152), 543–559.
- Rangel, A. (1997), Social security reform: Efficiency gains or intergenerational redistribution. Mimeo, Harvard University.
- Rebelo, S. (1991), 'Long-run policy analysis and long-run growth', *Journal of Political Economy* 99(3), 500–521.
- Rogerson, R. (1988), 'Indivisible labor, lotteries and equilibrium', *Journal of Monetary Economics* 21(1), 3–16.
- Rogerson, W. P. (1985), 'Repeated moral hazard', *Econometrica* 53(1), 69–76.
- Romer, P. M. (1986), 'Increasing returns and long-run growth', *Journal of Political Economy* 94(5), 1002–1037.
- Sachs, J. D. (1981), The current account and macroeconomic adjustment in the 1970s, in 'Brookings Papers on Economic Activity', Vol. 12, Brookings Institution, Washington, pp. 201–282.
- Samuelson, P. (1939), 'The gains from international trade', *Canadian Journal of Economics and Political Science* 5(2), 195–205.
- Samuelson, P. (1958), 'An exact consumption-loan model of interest with or without the social contrivance of money', *Journal of Political Economy* 66(6), 467–482.
- Samuelson, P. (1964), 'Theoretical notes on trade problems', *Review of Economics and Statistics* 46(2), 145–154.
- Sandmo, A. (1970), 'The effect of uncertainty on saving decisions', *Review of Economic Studies* 37(3), 353–360.
- Sargent, T. J. (1987), *Dynamic Macroeconomic Theory*, Harvard University Press, Cambridge, Massachusetts.
- Sargent, T. J. and Wallace, N. (1973), 'The stability of models of money and growth with perfect foresight', *Econometrica* 41(6), 1043–1048.

- Sargent, T. J. and Wallace, N. (1975), ‘Rational’ expectations, the optimal monetary instrument, and the optimal money supply rule’, *Journal of Political Economy* 83(2), 241–254.
- Sargent, T. J. and Wallace, N. (1981), ‘Some unpleasant monetarist arithmetic’, *Federal Reserve Bank of Minneapolis Quarterly Review* 5(3), 1–17.
- Schlicht, E. (2006), ‘A variant of Uzawa’s theorem’, *Economics Bulletin* 5(6), 1–5.
- Sharpe, W. F. (1964), ‘Capital asset prices: A theory of market equilibrium under conditions of risk’, *Journal of Finance* 19(3), 425–442.
- Shell, K. (1971), ‘Notes on the economics of infinity’, *Journal of Political Economy* 79(5), 1002–1011.
- Shimer, R. (2010), *Labor Markets and Business Cycles*, Princeton University Press, Princeton.
- Simon, C. P. and Blume, L. (1994), *Mathematics for Economists*, Norton & Company, New York.
- Sims, C. A. (1980), ‘Macroeconomics and reality’, *Econometrica* 48(1), 1–48.
- Sims, C. A. (1994), ‘A simple model for study of the determination of the price level and the interaction of monetary and fiscal policy’, *Economic Theory* 4(3), 381–399.
- Stokey, N. L. and Lucas, R. E. (1989), *Recursive Methods in Economic Dynamics*, Harvard University Press, Cambridge, Massachusetts.
- Straub, L. and Werning, I. (2014), Positive long run capital taxation: Chamley-judd revisited, Working Paper 20441, NBER, Cambridge, Massachusetts.
- Strotz, R. H. (1956), ‘Myopia and inconsistency in dynamic utility maximization’, *Review of Economic Studies* 23(3), 165–180.
- Svensson, L. E. O. and Razin, A. (1983), ‘The terms of trade and the current account: The Harberger-Laursen-Metzler effect’, *Journal of Political Economy* 91(1), 97–125.
- Tirole, J. (1982), ‘On the possibility of speculation under rational expectations’, *Econometrica* 50(5), 1163–1181.
- Tirole, J. (1985), ‘Asset bubbles and overlapping generations’, *Econometrica* 53(5), 1071–1100.
- Tirole, J. (2006), *The Theory of Corporate Finance*, Princeton University Press, Princeton.
- Tobin, J. (1969), ‘A general equilibrium approach to monetary theory’, *Journal of Money, Credit, and Banking* 1(1), 15–29.

- Uzawa, H. (1961), 'Neutral inventions and the stability of growth equilibrium', *Review of Economic Studies* **28**(2), 117–124.
- Vaughan, D. R. (1970), 'A non recursive algorithm solution for the discrete Ricatti equation', *IEEE Transactions on Automatic Control* **AC-15**, 597–599.
- Wallace, N. (1981), 'A Modigliani-Miller theorem for open-market operations', *American Economic Review* **71**(3), 267–274.
- Weil, P. (1989), 'The equity premium puzzle and the risk-free rate puzzle', *Journal of Monetary Economics* **24**, 401–421.
- Werning, I. (2003), Standard dynamic programming for Aiyagari, Marcet, Sargent and Seppälä. Mimeo, MIT, Cambridge, Massachusetts.
- Werning, I. (2007), 'Optimal fiscal policy with redistribution', *Quarterly Journal of Economics* **122**(3), 925–967.
- Woodford, M. (1995), 'Price level determinacy without control of a monetary aggregate', *Carnegie-Rochester Conference Series on Public Policy* **43**, 1–46.
- Yaari, M. E. (1965), 'Uncertain lifetime, life insurance, and the theory of the consumer', *Review of Economic Studies* **32**(2), 137–150.
- Zeldes, S. P. (1989a), 'Consumption and liquidity constraints: An empirical investigation', *Journal of Political Economy* **97**(2), 305–346.
- Zeldes, S. P. (1989b), 'Optimal consumption with stochastic income: Deviations from certainty equivalence', *Quarterly Journal of Economics* **104**(2), 275–298.
- Zhu, X. (1992), 'Optimal fiscal policy in a stochastic growth model', *Journal of Economic Theory* **58**(2), 250–289.